Universidade de Aveiro Departamento de Matemática
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Isabel Margarida da Costa Andrade Xarez

Reflexões de álgebras universais em semi-reticulados, suas teorias de Galois e sistemas de factorização associados

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Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica do Doutor George Janelidze, Professor Catedrático no Department of Mathematics and Applied Mathematics da University of Cape Town, e do Doutor João Xarez, Professor Auxiliar do Departamento de Matemática da Universidade de Aveiro.

Dedico este trabalho aos meus filhos Simão, Manuel, José e Branca Maria.

o júri

presidente

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palavras-chave

Álgebras universais, variedades, subvariedades idempotentes, semi-reticulados, reflexões admissíveis, teorias de Galois.

resumo

Estabelecemos uma condição suficiente para a preservação dos produtos finitos, pelo reflector de uma variedade de álgebras universais numa subvariedade, que é, também, condição necessária se a subvariedade for idempotente. Esta condição é estabelecida, seguidamente, num contexto mais geral e caracteriza reflexões para as quais a propriedade de ser semi-exacta à esquerda e a propriedade, mais forte, de ter unidades estáveis, coincidem.
Prova-se que reflexões simples e semi-exactas à esquerda coincidem, no contexto das variedades de álgebras universais e caracterizam-se as classes do sistema de factorização derivado da reflexão. Estabelecem-se resultados que ajudam a caracterizar morfismos de cobertura e verticais-estáveis em álgebras universais e no contexto mais geral já referido.
Caracterizam-se as classes de morfismos separáveis, puramente inseparáveis e normais. O estudo dos morfismos de descida de Galois conduz a condições suficientes para que o seu par kernel seja preservado pelo reflector.

keywords

Universal algebras, varieties, idempotent subvarieties, semilattices, admissible reflections, Galois theories.

abstract

We begin with a sufficient condition for the preservation of finite products by a reflector from a variety of universal algebras into a subvariety, which is also a necessary condition when the subvariety is idempotent. This condition is then stated in a more general setting and this characterizes reflections for which semileft-exactness and the stronger stable units property are the same.
It is shown that simple and semi-left-exact reflections coincide in the context of varieties of universal algebras, and characterizations of the classes of the derived reflective factorization system are given. Several statements help then to characterize covering and stably-vertical morphisms of universal algebras, and in the more general setting referred to above. The classes of separable, purely inseparable and normal morphisms are characterized as well. The study of Galois descent morphisms provides conditions under which their kernel pairs are preserved by the reflector.

Contents

Introduction

Semi-left-exact reflections and reflections with stable units were originally introduced by C. Cassidy, M. Hébert, and G. M. Kelly [**3**] as reflections that preserve certain pullbacks. In [**13**], an additional structure on a reflection $I : \mathbb{C} \to \mathbb{M}$ was described, involving a pullback-preserving functor $U : \mathbb{C} \to \mathbf{Set}$, which allowed to simplify these preservation conditions by reducing them to preservation of very special pullbacks. These results were applied to reflections of (i) varieties of universal algebras into subvarieties of idempotent algebras; (ii) (the category of) compact Hausdorff spaces into Stone spaces.

Categorical version of monotone-light factorization for continuous maps of compact Hausdorff spaces was obtained by A. Carboni, G. Janelidze, G. M. Kelly, and R. Paré in [**2**]. The results on the reflection of semigroups into semillatices obtained by G. Janelidze, V. Lann, and L. Màrki in [**9**] look similar to the results on the reflection of compact Hausdorff spaces into Stone spaces. In [**13**], J. J. Xarez showed that this is not similarity, but two special cases of the same 'theory'. See also [**15**], where these results were generalized and new examples presented.

The author of [**13**] mentioned that the results in that paper had their origin in generalizing the proof of Theorem 3 in the article [**9**] of Janelidze, G., Lann, V., Márki, L., where, among other results, they conclude that the reflection of the variety **SGr** of semigroups into the subvariety **SLat** of semilattices has stable units and, therefore, preserves finite products. In fact, the other way around is valid in the setting of [**13**], that is, the preservation of finite products implies that if a reflection is semi-left-exact then it has also the stronger property of having stable units.

A reflection I from a category $\mathbb{C}$ into its full subcategory $\mathbb{M}$ can be seen as a Galois structure, one in which all morphisms are taken into account. Hence, such a reflection is semi-left-exact (in the sense of [**3**]) if and only if it is an admissible Galois structure (in the sense of categorical Galois theory). Semi-left-exactness was called attainability in [**12**], in the particular case of semigroups.

The "state of the art" having been described in the preceding paragraphs, we are now going to give an account of our contributions, as presented in the current work.

Our work begins by applying the theory above to (semigroups again and) universal algebras.

We state a necessary and sufficient condition for the preservation of finite products by the reflection $I : \mathbb{C} \to \mathbb{M}$ from a variety $\mathbb{C}$ of universal algebras into an idempotent subvariety $\mathbb{M}$; for instance, this applies to the reflection of the variety of semigroups into the variety **Band** of bands (idempotent semigroups). We also conclude that when $\mathbb{C}$ is idempotent, that is, the free algebra on a one-element set itself has only one element, finite products are always preserved. In particular, it is the case for the reflection of bands into semilattices (idempotent and commutative semigroups). Under those conditions, the reflection I is semi-left-exact if and only if it has stable units.

We characterize the factorization system $(\mathcal{E}_I, \mathcal{M}_I)$ derived from a simple reflection of a variety of universal algebras into a subvariety in it.

We conclude that a reflection of a variety of universal algebras into a subvariety is simple if and only if it is semi-left-exact.

We characterize the class $\mathcal{E}'_I$ of stably-vertical morphisms for some reflections $I : \mathbb{C} \to \mathbb{M}$ into idempotent subvarieties. For instance, $\mathcal{E}'_I = \mathcal{E}_I \cap \mathcal{E}$ for the reflection $\mathbf{Band} \to \mathbf{SLat}$, of bands into semilattices, where $\mathcal{E}$ denotes the class of surjective homomorphisms. On the other hand, for the reflection $\mathbf{CommSgr} \to \mathbf{SLat}$, of commutative semigroups into semilattices, $\mathcal{E}'_I = \mathcal{E}_I \cap \mathcal{F}$, where $\mathcal{F} = \{e : X \to Y \in \mathbf{CommSgr} \mid \forall_{y \in Y}\ \langle y \rangle_Y \cap e(X) \neq \emptyset\}$. Here $\langle y \rangle_Y$ denotes the subalgebra of Y generated by $\{y\}$.

We state sufficient conditions for the coincidence of the category $\mathcal{M}^*_I/B$ of coverings of B with the category $\mathcal{M}_I/B$ of trivial coverings of B, for a simple reflection $I : \mathbb{C} \to \mathbb{M}$ into an idempotent subvariety. For instance, $\mathcal{M}^*_I = \mathcal{M}_I$ for the reflection of bands into semilattices, and for the reflection of commutative semigroups into semilattices.

Under these conditions the kernel pair of a Galois descent morphism $\sigma : S \to R$ is preserved by the reflector.

We also describe other classes of morphisms that occur in Galois theory, namely the classes of separable, purely inseparable, and normal morphisms. Under certain additional conditions those are the homomorphisms $\alpha : S \to R$ such that, respectively,

- $Ker(\alpha) \cap \sim_S = \Delta$,
- $Ker(\alpha) \subseteq \sim_S$,
- $\sim_S \circ Ker(\alpha) \subseteq Ker(\alpha) \circ \sim_S$ and $Ker(\alpha) \cap \sim_S = \Delta$,

where Δ denotes the equality relation, $Ker(\alpha)$ denotes the kernel pair of α, $\sim_S$ denotes the congruence on S induced by the reflection and $\circ$ denotes the composition of congruences. For example this applies to both reflections of bands into semilattices and of commutative semigroups into semilattices.

We conclude that if the class $\mathcal{E}'_I$ of stably-vertical morphisms is just $\mathcal{E}_I \cap \mathcal{F}$, where $\mathcal{F} = \{e : X \to Y \in \mathbb{C} \mid \forall_{y \in Y}\ \langle y \rangle_Y \cap e(X) \neq \emptyset\}$, then there is an inseparable-separable factorization system. For instance, this is the case for both reflections of bands into semilattices and of commutative semigroups into semilattices. Although none of these two reflections has a monotone-light factorization system.

Many of the results described above will be generalized to abstract reflections $I : \mathbb{A} \to \mathbb{B}$, from a finitely complete category $\mathbb{A}$ into a full subcategory $\mathbb{B}$, provided there exists a functor $U : \mathbb{A} \to \mathbf{Set}$ which preserves finite products and reflects isomorphisms.

CHAPTER 1

Preliminary notions

In this chapter, those aspects of factorization systems and categorical Galois theory referred to in next chapters are presented.

The subjects of the following sections 1.1 and 1.2 can be found in detail in [**2**] and [**3**], the latter reference being the source of the considered notions.

1.1. Factorization systems

Let $\mathbb{C}$ be a category and let f, g be morphisms in $\mathbb{C}$. We write $f \downarrow g$ if, for every pair of morphisms u, v in $\mathbb{C}$ with $vf = gu$ there is a unique w making commutative the following diagram.

(1.1)
$$\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "v"'] \arrow[ur, "w"] & \bullet
\end{tikzcd}$$

For any class $\mathcal{H}$ of morphisms in $\mathbb{C}$ we set

$$\mathcal{H}^{\uparrow} = \{f \mid f \downarrow h \ for\ all\ h \in \mathcal{H}\},$$

$$\mathcal{H}^{\downarrow} = \{g \mid h \downarrow g \ for\ all\ h \in \mathcal{H}\}.$$

A prefactorization system is a pair of classes of morphisms $(\mathcal{E}, \mathcal{M})$ having $\mathcal{E} = \mathcal{M}^{\uparrow}$ and $\mathcal{M} = \mathcal{E}^{\downarrow}$.

Proposition 1.1. *Let $(\mathcal{E}, \mathcal{M})$ be a prefactorization system. Then:*

(a) *$\mathcal{M}$ contains the isomorphisms and is closed under composition;*

(b) *every pullback of a morphism in* $\mathcal{M}$ *is in* $\mathcal{M}$;

(c) *if* $f \circ g$ *is in* $\mathcal{M}$, *so is* g, *provided that* f *is either a monomorphism or is in* $\mathcal{M}$;

(d) *the intersection* $\mathcal{E} \cap \mathcal{M}$ *consists of the isomorphisms.*

A pair of classes of morphisms in $\mathbb{C}$ is said to constitute a factorization system if

(i) each of $\mathcal{E}$ and $\mathcal{M}$ contains the identities and is closed under composition with isomorphisms;

(ii) every morphism f in $\mathbb{C}$ can be written as $f = m \circ e$, with $m \in \mathcal{M}$ and $e \in \mathcal{E}$;

(iii) if $v \circ m \circ e = m' \circ e' \circ u$ with $m,\ m' \in \mathcal{M}$ and $e,\ e' \in \mathcal{E}$, there is a unique w making commutative the following diagram,

(1.2)
$$\begin{tikzcd}
\cdot \arrow[r, "e"] \arrow[d, "u"] & \cdot \arrow[r, "m"] \arrow[d, "w"] & \cdot \arrow[d, "v"] \\
\cdot \arrow[r, "e'"'] & \cdot \arrow[r, "m'"'] & \cdot
\end{tikzcd}.$$

PROPOSITION 1.2. *Every factorization system is a prefactorization system; in particular,* $\mathcal{E}$ *and* $\mathcal{M}$ *are both closed under composition.*

PROPOSITION 1.3. *Factorization systems are just those prefactorization systems that satisfy* (ii) *above.*

1.2. Factorization systems derived from a reflection

Let $H \vdash I : \mathbb{C} \to \mathbb{M}$ be a reflection of a finitely complete category $\mathbb{C}$ into a full subcategory $\mathbb{M}$, with unit morphism $\eta : \mathbf{1}_{\mathbb{C}} \to HI$ and let $(\mathcal{E}_I, \mathcal{M}_I)$ be a prefactorization system as in [**2**, §3], that is:

$$\mathcal{E}_I = (H(mor\mathbb{M}))^{\uparrow}, \quad \mathcal{M}_I = (H(mor\mathbb{M}))^{\uparrow\downarrow},$$

where $H(mor\mathbb{M})$ stands for the class of all morphisms in $\mathbb{C}$ which belong to the subcategory $\mathbb{M}$.

A morphism $e : A \to B$ in $\mathbb{C}$ belongs to $\mathcal{E}_I$ if and only if $I(e)$ is an isomorphism. Hence, if $e \in \mathcal{E}_I$ and $e \circ f \in \mathcal{E}_I$ then $f \in \mathcal{E}_I$. In particular, since the subcategory $\mathbb{M}$ is full, $\eta_C : C \to HI(C)$ lies in $\mathcal{E}_I$. Every morphism in $\mathbb{M}$ lies in $\mathcal{M}_I$.

Definition 1.4. The reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ is called simple if $w \in \mathcal{E}_I$ in every diagram of the form

(1.3)
$$\begin{tikzcd}
A \arrow[dr, "w"] \arrow[drr, "\eta_A"] \arrow[ddr, "f"'] & & \\
& B \times_{HI(B)} HI(A) \arrow[r] \arrow[d, "\pi_1"] & HI(A) \arrow[d, "HI(f)"] \\
& B \arrow[r, "\eta_B"] & HI(B)
\end{tikzcd} ,$$

where the rectangular part of the diagram is a pullback square, η_A and η_B are unit morphisms, and w is the unique morphism which makes the diagram commute.

Hence, $(\mathcal{E}_I, \mathcal{M}_I)$ is a factorization system if the reflection is simple, since π_1 in diagram (1.3) is a pullback of a morphism in $\mathcal{M}_I$, and so it is in $\mathcal{M}_I$.

Proposition 1.5. *The following conditions are equivalent:*

(1) *the reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ is simple;*

(2) *a map $f : A \to B$ is in $\mathcal{M}_I$ if and only if the following diagram is a pullback,*

(1.4)
$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "f"'] & HI(A) \arrow[d, "HI(f)"] \\
B \arrow[r, "\eta_B"] & HI(B)
\end{tikzcd} ,$$

where η_A and η_B are unit morphisms.

DEFINITION 1.6. The reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ is semi-left-exact (also called admissible) if $\pi_2 \in \mathcal{E}_I$ in every pullback diagram of the form

$$\begin{tikzcd} C \times_{HI(C)} M \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & M \arrow[d, "g"] \\ C \arrow[r, "\eta_C"] & HI(C), \end{tikzcd} \tag{1.5}$$

where η_C is a unit morphism and $M \in \mathbb{M}$.

PROPOSITION 1.7. *The reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ is semi-left-exact if and only if any of the two following conditions holds:*

(1) *every pullback of a morphism in $\mathcal{M}_I$ is preserved by the left adjoint I;*

(2) *every pullback of a morphism in $\mathcal{E}_I$ along a morphism in $\mathcal{M}_I$ is in $\mathcal{E}_I$.*

If a reflection is semi-left-exact then it is simple, but the converse is not true in general.

DEFINITION 1.8. A reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ has stable units if the reflector I preserves all pullback diagrams of the form

$$\begin{tikzcd} C \times_{HI(C)} D \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & D \arrow[d, "g"] \\ C \arrow[r, "\eta_C"] & HI(C), \end{tikzcd} \tag{1.6}$$

where η_C is a unit morphism.

If a reflection has stable units then it is semi-left-exact, but the converse is not true in general.

1.3. Connected components, semi-left-exactness, stable units

The following two Lemmas 1.10 and 1.11 are stated and proved in [**13**] (see also [**15**]).

Assume the following data (1.7):

(1) A reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ of a finitely complete category $\mathbb{C}$ into its full subcategory $\mathbb{M}$, with unit $\eta : 1_{\mathbb{C}} \to HI$.

(2) A functor $U : \mathbb{C} \to \mathbf{Set}$, such that:

   (a) U preserves finite limits;

   (b) $UH : \mathbb{M} \to \mathbf{Set}$ reflects isomorphisms;

   (c) every map $U(\eta_C) : U(C) \to UHI(C)$ is a surjection, for every unit morphism $\eta_C, C \in \mathbb{C}$;

   (d) every map $U_{T,M} : \mathbb{C}(T, M) \to \mathbf{Set}(\{*\}, U(M))$ is a surjection, for any object $M \in \mathbb{M}$, with T a terminal object in $\mathbb{C}$.

Definition 1.9. Consider any morphism $\mu : T \to HI(C)$ from a terminal object T into $HI(C)$, for some $C \in \mathbb{C}$. The connected component associated to the morphism μ is the pullback C_μ in the following pullback square.

(1.8)
$$\begin{tikzcd} C_\mu \arrow[r] \arrow[d] & T \arrow[d, "\mu"] \\ C \arrow[r, "\eta_C"] & HI(C) \end{tikzcd}$$

Lemma 1.10. *Under data* (1.7), (1) *and* (2),

$I \dashv H$ is semi-left-exact
if and only if
$HI(C_\mu) \cong T$

for every connected component C_μ,

where T *is any terminal object.*

LEMMA 1.11. *Under data* (1.7), (1) *and* (2),

$$\begin{array}{l} I \dashv H \textit{ has stable units} \\ \quad \textit{if and only if} \\ \quad HI(C_\mu \times D_\nu) \cong T \\ \textit{for every pair of connected components } C_\mu \textit{ and } D_\nu, \end{array}$$

where T *is any terminal object.*

1.4. Aspects of categorical Galois theory

Categorical Galois theory is presented in [**1**]. Here we review some results concerning semi-left-exact reflections. Some other results concerning monadic functors (cf. [**11**]) are needed in order to define effective descent morphisms.

Internal groupoid :

DEFINITION 1.12. Let $\mathbb{A}$ be a category with pullbacks. An internal category $\mathbb{I}$ in $\mathbb{A}$ consists in giving the following diagram in $\mathbb{A}$,

$$\begin{tikzcd} C_2 \arrow[r, shift left=2, "f_0"] \arrow[r, "m"] \arrow[r, shift right=2, "f_1"'] & C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & C_0 \arrow[l, "n"'] \end{tikzcd} \tag{1.9}$$

where

- C_0 is the object of objects,
- C_1 is the object of morphisms,
- C_2 is the object of pairs of composable morphisms,

- d_0 is the domain morphism,
- d_1 is the codomain morphism,
- $n : C_0 \to C_1$ is the identity morphism ($c \mapsto id_C$ if $\mathbb{A}$ is **Set**),
- $m : C_2 \to C_1$ is the composition morphism ($\langle g,\ f\rangle \mapsto g \circ f$ if $\mathbb{A}$ is **Set**),

subject to the following axioms:

($\mathbf{G_1}$) the square

$$\begin{tikzcd} C_2 \arrow[r, "f_0"] \arrow[d, "f_1"] & C_1 \arrow[d, "d_1"] \\ C_1 \arrow[r, "d_0"] & C_0 \end{tikzcd} \tag{1.10}$$

is a pullback, which defines C_2 as the object of composable morphisms of C_1;

($\mathbf{G_2}$) the triangles

$$\begin{tikzcd} C_0 \arrow[r, "n"] \arrow[dr, equal] & C_1 \arrow[d, "d_0"] \\ & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_0 \arrow[r, "n"] \arrow[dr, equal] & C_1 \arrow[d, "d_1"] \\ & C_0 \end{tikzcd} \tag{1.11}$$

are commutative, which express, in case $\mathbb{A}$ is **Set**, that $d_0(id_a) = a = d_1(id_a)$, for every object $a \in C_0$;

($\mathbf{G_3}$) the squares

(1.12)
$$\begin{tikzcd} C_2 \arrow[r, "m"] \arrow[d, "f_0"] & C_1 \arrow[d, "d_0"] \\ C_1 \arrow[r, "d_0"] & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_2 \arrow[r, "m"] \arrow[d, "f_1"] & C_1 \arrow[d, "d_1"] \\ C_1 \arrow[r, "d_1"] & C_0 \end{tikzcd}$$

are commutative, which express, in case $\mathbb{A}$ is **Set**, $d_0(g \circ f) = d_0(f)$, $d_1(g \circ f) = d_1(g)$, for all composable pairs (g, f) ;

($\mathbf{G_4}$) the triangles

(1.13)
$$\begin{tikzcd} C_1 \arrow[r, "\langle id_{C_1},\ nd_0 \rangle"] \arrow[dr, equal] & C_2 \arrow[d, "m"] \\ & C_1 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "\langle nd_1,\ id_{C_1} \rangle"] \arrow[dr, equal] & C_2 \arrow[d, "m"] \\ & C_1 \end{tikzcd}$$

are commutative, which express the unit law;

($\mathbf{G_5}$) considering the pullback

(1.14)
$$\begin{tikzcd} C_3 \arrow[r, "g_0"] \arrow[d, "g_1"] & C_2 \arrow[d, "f_1"] \\ C_2 \arrow[r, "f_0"] & C_1 \end{tikzcd}$$

the diagram

(1.15)
$$\begin{tikzcd} C_3 \arrow[r, "\langle m \circ g_1,\ f_0 \circ g_0 \rangle"] \arrow[d, "\langle f_1 \circ g_1,\ m \circ g_0 \rangle"'] & C_2 \arrow[d, "m"] \\ C_2 \arrow[r, "m"] & C_1 \end{tikzcd}$$

is commutative, which express the associative law.

An internal groupoid $\mathbb{G}$ is an internal category $\mathbb{I}$ as above, together with an additional datum, namely, a morphism

$$\tau : C_1 \to C_1$$

that formally inverts any morphism in C_1, that is, τ is such that:

($\mathbf{G_6}$) the triangles

$$\begin{tikzcd} C_1 \arrow[r, "\tau"] \arrow[dr, "d_1"'] & C_1 \arrow[d, "d_0"] \\ & C_0 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "\tau"] \arrow[dr, "d_0"'] & C_1 \arrow[d, "d_1"] \\ & C_0 \end{tikzcd} \tag{1.16}$$

are commutative;

($\mathbf{G_7}$) the squares

$$\begin{tikzcd} C_1 \arrow[r, "{\langle \tau,\ id_{C_1} \rangle}"] \arrow[d, "d_0"] & C_2 \arrow[d, "m"] \\ C_0 \arrow[r, "n"] & C_1 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r, "{\langle id_{C_1},\ \tau \rangle}"] \arrow[d, "d_1"] & C_2 \arrow[d, "m"] \\ C_0 \arrow[r, "n"] & C_1 \end{tikzcd} \tag{1.17}$$

are commutative.

PROPOSITION 1.13. *Let $\sigma : S \to R$ be a morphism in a category $\mathbb{A}$ with pullbacks. The following diagram*

$$\begin{tikzcd} (S \times_R S) \times_S (S \times_R S) \arrow[r, shift left=2, "f_0"] \arrow[r, "m"] \arrow[r, shift right=2, "f_1"'] & S \times_R S \arrow[r, shift left=2, "\pi_2"] \arrow[r, shift right=2, "\pi_1"'] & S \arrow[l, "\Delta"'] \end{tikzcd} \tag{1.18}$$

is an internal groupoid, with:

- $(\pi_1,\ \pi_2)$ *the kernel pair of* σ

$$
\begin{tikzcd}
S \times_R S \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & S \arrow[d, "\sigma"] \\
S \arrow[r, "\sigma"] & R \ ;
\end{tikzcd}
\tag{1.19}
$$

- $\Delta : S \to S \times_R S$ *is the diagonal in the commutative diagram*

$$
\begin{tikzcd}
S \arrow[dr, "\Delta"] \arrow[drr, "id_S"] \arrow[ddr, "id_S"'] & & \\
& S \times_R S \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & S \arrow[d, "\sigma"] \\
& S \arrow[r, "\sigma"] & R \ ;
\end{tikzcd}
\tag{1.20}
$$

- $\tau : S \times_R S \to S \times_R S$ *is the twisting isomorphism which interchanges factors, given in the commutative diagram*

$$
\begin{tikzcd}
S \times_R S \arrow[dr, "\tau"] \arrow[drr, "\pi_1"] \arrow[ddr, "\pi_2"'] & & \\
& S \times_R S \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & S \arrow[d, "\sigma"] \\
& S \arrow[r, "\sigma"] & R \ ;
\end{tikzcd}
\tag{1.21}
$$

- $f_0,\ f_1 : (S \times_R S) \times_S (S \times_R S) \to S \times_R S$ *are the morphisms in the following pullback diagram*

$$
\begin{tikzcd}
(S \times_R S) \times_S (S \times_R S) \arrow[r, "f_0"] \arrow[d, "f_1"] & S \times_R S \arrow[d, "\pi_1"] \\
S \times_R S \arrow[r, "\pi_2"] & S \quad ;
\end{tikzcd}
\tag{1.22}
$$

• *the following pullback diagram defines*

$$m : (S \times_R S) \times_S (S \times_R S) \to S \times_R S,$$

$$
\begin{tikzcd}
(S \times_R S) \times_S (S \times_R S) \arrow[dr, "m"] \arrow[drr, "\pi_2 \circ f_0"] \arrow[ddr, "\pi_1 \circ f_1"'] & & \\
& S \times_R S \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & S \arrow[d, "\sigma"] \\
& S \arrow[r, "\sigma"] & R \quad ;
\end{tikzcd}
\tag{1.23}
$$

the following diagram, where all squares are pullbacks, shows that the outside square of diagram (1.23) *is commutative,*

$$
\begin{tikzcd}
{} \arrow[r, "f_0"] \arrow[d, "f_1"'] & S \times_R S \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & S \arrow[d, "\sigma"] \\
S \times_R S \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"'] & S \arrow[r, "\sigma"'] \arrow[d, "\sigma"] & R \\
S \arrow[r, "\sigma"'] & R & .
\end{tikzcd}
\tag{1.24}
$$

It is obvious that the kernel pair of a morphism besides being an internal groupoid is, as well, an equivalence relation.

Definition 1.14. Let $\mathbb{C}$ be a category. For an object $C \in \mathbb{C}$, we write $\mathbb{C}/C$ for the following category:

(1) the objects are the pairs $(X,\ f)$ where $f : X \to C$;

(2) the morphisms $h : (X,\ f) \to (X',\ f')$ are all morphisms $h : X \to X'$ in $\mathbb{C}$ such that $f' \circ h = f$.

Lemma 1.15. *Consider an adjunction $G \vdash F : \mathbb{A} \to \mathbb{B}$. Assume that $\mathbb{A}$ has pullbacks. Then, for every object $A \in \mathbb{A}$, the functor*

$$F_A : \mathbb{A}/A \to \mathbb{B}/F(A), \quad (X,\ \alpha) \mapsto (F(X),\ F(\alpha)),$$

has a right adjoint functor

$$G_A : \mathbb{B}/F(A) \to \mathbb{A}/A, \quad (Y,\ \beta) \mapsto (Z,\ \gamma),$$

where $(Z,\ \gamma)$ is given in the pullback

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "\gamma"'] & G(Y) \arrow[d, "G(\beta)"] \\ A \arrow[r, "\eta_A"] & GF(A) \end{tikzcd} \tag{1.25}$$

where η_A is a unit morphism.

Proposition 1.16. *Let $\mathbb{C}$ be a category with pullbacks and $H \vdash I : \mathbb{C} \to \mathbb{M}$ a reflection of $\mathbb{C}$ into a full subcategory $\mathbb{M}$. The reflection is semi-left-exact when, for every object $C \in \mathbb{C}$, the counit $\varepsilon^C : I_C H_C \to \mathbf{1}_{\mathbb{M}/I(C)}$ of the adjunction*

$$H_C \vdash I_C : \mathbb{C}/C \to \mathbb{M}/I(C),$$

given by Lemma 1.15, is an isomorphism, that is, H_C is a fully faithful functor.

Proof. We have to prove that the reflection is semi-left-exact if and only if the counit morphisms of the adjunction $H_C \vdash I_C : \mathbb{C}/C \to \mathbb{M}/I(C)$, $\varepsilon^C_{(X,\ f)} : I_C H_C(X,\ f) \to (X,\ f)$, are isomorphisms for every $X \in \mathbb{M},\ f : X \to I(C)$.

Consider the following pullback diagram where $H_C(X,\ f) = (P,\ \pi_1)$,

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & H(X) \arrow[d, "H(f)"] \\ C \arrow[r, "\eta_C"] & HI(C) \end{tikzcd} , \tag{1.26}$$

and the following commutative diagram, where η and ε are respectively the unit and counit of the reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$.

$$\begin{tikzcd} I(P) \arrow[r, "I(\pi_2)"] \arrow[d, "I(\pi_1)"'] & IH(X) \arrow[r, "\varepsilon_X"] \arrow[d, "IH(f)"] & X \arrow[d, "f"] \\ I(C) \arrow[r, "I(\eta_C)"'] & IHI(C) \arrow[r, "\varepsilon_{I(C)}"'] & I(C) \end{tikzcd} \tag{1.27}$$

In the previous diagram, $(I(P),\ I(\pi_1))$ is the object $I_C H_C(X,\ f) \in \mathbb{M}/I(C)$ and $\varepsilon_{I(C)} \circ I(\eta_C) = \mathbf{1}_{I(C)}$. On the other hand, $\varepsilon_X \circ I(\pi_2)$ is universal from I_C to $(X,\ f)$. Hence $\varepsilon_X \circ I(\pi_2) = \varepsilon_{(X,\ f)} : I_C H_C(X,\ f) \to (X,\ f)$ is an isomorphism if and only if $I(\pi_2)$ is an isomorphism, that is, if and only if $\pi_2 \in \mathcal{E}_I$, since H is fully-faithful.

$\square$

Definition 1.17. A monad $T = \langle T, \eta, \mu \rangle$ in a category $\mathbb{A}$ consists of a functor $T : \mathbb{A} \to \mathbb{A}$ and two natural transformations

$$\eta : \mathbf{1}_{\mathbb{A}} \to T, \qquad \mu : T^2 \to T$$

which make the following diagrams commute

$$
\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"] & T
\end{tikzcd}
,\qquad
\begin{tikzcd}
\mathbf{1}_{\mathbb{A}}T \arrow[r, "\eta T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T\mathbf{1}_{\mathbb{A}} \arrow[l, "T\eta"'] \arrow[dl, equal] \\
& T &
\end{tikzcd}
. \tag{1.28}
$$

Every adjunction $\langle F, G, \eta, \varepsilon \rangle : \mathbb{A} \to \mathbb{B}$ gives rise to a monad $\langle GF, \eta, G\varepsilon F \rangle$ in the category $\mathbb{A}$, where $T = GF$, $\eta : \mathbf{1}_{\mathbb{A}} \to T$, and $\mu = G\varepsilon F : GFGF \to GF = T$.

Definition 1.18. If $T = \langle T, \eta, \mu \rangle$ is a monad in $\mathbb{A}$, a T-algebra $\langle A, h \rangle$ is a pair consisting of an object $A \in \mathbb{A}$ (the underlying object of the algebra) and an arrow $h : TA \to A$ of $\mathbb{A}$ (called the structure map of the algebra) which makes both next diagrams commute (the first diagram is the associative law and the second is the unit law),

$$
\begin{tikzcd}
T^2A \arrow[r, "Th"] \arrow[d, "\mu_A"'] & TA \arrow[d, "h"] \\
TA \arrow[r, "h"] & A
\end{tikzcd}
,\qquad
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "\mathbf{1}_A"'] & TA \arrow[d, "h"] \\
& A
\end{tikzcd}
. \tag{1.29}
$$

A morphism $f : \langle A, h \rangle \to \langle A', h' \rangle$ of T-algebras is an arrow $f : A \to A'$ which makes the following diagram commute,

$$
\begin{tikzcd}
TA \arrow[r, "h"] \arrow[d, "Tf"'] & A \arrow[d, "f"] \\
TA' \arrow[r, "h'"] & A'
\end{tikzcd}
. \tag{1.30}
$$

Theorem 1.19. *If $\langle T, \eta, \mu \rangle$ is a monad in $\mathbb{A}$, then the set of all T-algebras and their morphisms form a category $\mathbb{A}^T$. There is an adjunction*

$$\langle F^T, G^T; \eta^T, \varepsilon^T\rangle : \mathbb{A} \to \mathbb{A}^T$$

in which the functors G^T and F^T are given by the respective following assignments

$$\begin{tikzcd} \langle A,h\rangle \arrow[r] \arrow[d, "f"] & A \arrow[d, "f"] \\ \langle A',h'\rangle \arrow[r] & A' \end{tikzcd}, \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "f"] & \langle TA,\mu_A\rangle \arrow[d, "f"] \\ A' \arrow[r] & \langle TA',\mu'_A\rangle \end{tikzcd}, \tag{1.31}$$

with $G^T :$ the left assignment and $F^T :$ the right assignment,

and $\eta^T = \eta$ and $\varepsilon^T\langle A,h\rangle = h$, for each T-algebra $\langle A,h\rangle$. The monad defined in $\mathbb{A}$ by this adjunction is the given monad $\langle T,\eta,\mu\rangle$.

THEOREM 1.20. *Let $\langle F,G,\eta,\varepsilon\rangle : \mathbb{A} \to \mathbb{B}$ be an adjunction, $T = \langle GF, \eta,\ G\varepsilon F\rangle$ the monad it defines in $\mathbb{A}$. Then, there is a unique functor $K : \mathbb{B} \to \mathbb{A}^T$ with $G^T K = G$ and $KF = F^T$.*

DEFINITION 1.21. Let $\langle F,G,\eta,\ \varepsilon\rangle : \mathbb{A} \to \mathbb{B}$ be an adjunction and $K : \mathbb{B} \to \mathbb{A}^T$ be the functor in Theorem 1.20.

The right adjoint $G : \mathbb{B} \to \mathbb{A}$ is called monadic if K is a category equivalence.

PROPOSITION 1.22. *Let $f : S \to R$ be a morphism in a category $\mathbb{A}$.*

The functor "pullback along f"

$$f^* : \mathbb{A}/R \to \mathbb{A}/S$$

admits the left adjoint functor

$$\Sigma_f : \mathbb{A}/S \to \mathbb{A}/R, \qquad (A,a) \mapsto (A, f\circ a)$$

of composition with f.

DEFINITION 1.23. Let $\mathbb{A}$ be a category with pullbacks. A morphism $f : S \to R$ in $\mathbb{A}$ is an effective descent morphism when the functor "pullback along f"

$$f^* : \mathbb{A}/R \to \mathbb{A}/S$$

is monadic.

PROPOSITION 1.24. *The functor f^* in Definition 1.23 is monadic if and only if the following conditions hold:*

(1) *the functor f^* reflects isomorphisms;*

(2) *the functor f^* creates the coequalizers of those pairs (u, v) such that $(f^*(u), f^*(v))$ has a split coequalizer.*

A split coequalizer of $(f^*(u), f^*(v))$ consists in three morphisms q, r and s, such that $q \circ f^*(u) = q \circ f^*(v)$, $q \circ r = id$, $f^*(u) \circ s = id$ and $f^*(v)s = rq$; condition (2) *requires that for each such pair (u, v), the coequalizer of (u, v) exists in $\mathbb{A}/R$ and is preserved by f^* (cf.* [**1**]*).*

It is well known that, in the context of varieties of universal algebras, effective descent morphisms are just the surjective homomorphisms.

Definition 1.25. Consider a semi-left-exact reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ of a category $\mathbb{C}$ with pullbacks into a full subcategory $\mathbb{M}$.

An object $(A,\ a) \in \mathbb{C}/R$ is split by a morphism $\sigma : S \to R$ of $\mathbb{C}$, when the unit

$$\eta^S_{\sigma^*(A,\ a)} : \sigma^*(A,\ a) \to H_S I_S(\sigma^*(A,\ a))$$

of the reflection $H_S \vdash I_S$ is an isomorphism, where $\sigma^*(A,\ a)$ denotes the pullback of $(A,\ a)$ along σ.

The class of objects $(A,\ a) \in \mathbb{C}/R$ which are split by σ determines the full subcategory $\mathbf{Split}_R(\sigma)$ of $\mathbb{C}/R$.

Proposition 1.26. *Under the conditions of Definition* 1.25 *an object $(A,\ a) \in \mathbb{C}/R$ is split by a morphism $\sigma : S \to R$ in $\mathbb{C}$ if and only if $\sigma^*(A,\ a) \in \mathcal{M}_I$ (cf. section 1.2).*

Proof. Consider the following commutative diagram, where the left square is a pullback and $H_S I_S(\sigma^*(A,\ a)) = (P^*,\ \pi_1)$.

$$\begin{tikzcd}
P \arrow[dr, "\eta^S_{\sigma^*(A,\ a)}"] \arrow[drr, "\eta_P"] \arrow[ddr, "\sigma^*(A,\ a)"'] & & & \\
& P^* \arrow[r] \arrow[d, "\pi_1"] & HI(P) \arrow[r] \arrow[d, "HI(\sigma^*(A,\ a))"] & HI(A) \arrow[d, "HI(a)"] \\
& S \arrow[r, "\eta_S"'] & HI(S) \arrow[r, "HI(\sigma)"'] & HI(R)
\end{tikzcd} \tag{1.32}$$

Since the reflection is semi-left-exact, and therefore also a simple reflection (cf. Proposition 1.5), $\eta^S_{\sigma^*(A,\ a)}$ is an isomorphism if and only if $\sigma^*(A,\ a) \in \mathcal{M}_I$.

□

DEFINITION 1.27. Let us consider the following data:

- a category $\mathbb{C}$ with pullbacks;
- a semi-left-exact reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$;
- a morphism $\sigma : S \to R$ in $\mathbb{C}$.

A morphism σ is called of Galois descent, with respect to the semi-left-exact reflection $H \vdash I$, if the following two conditions hold:

(1) σ is an effective descent morphism in $\mathbb{C}$;

(2) σ is split by itself, i.e., $\pi_1 \in \mathcal{M}_I$ in the following pullback,

(1.33)
$$\begin{tikzcd} S \times_R S \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & S \arrow[d, "\sigma"] \\ S \arrow[r, "\sigma"] & R \end{tikzcd} .$$

LEMMA 1.28. *In the conditions of Definition 1.27, the functor* $I : \mathbb{C} \to \mathbb{M}$ *transforms the kernel pair of* $\sigma : S \to R$, *seen as an internal groupoid in* $\mathbb{C}$, *into an internal groupoid in* $\mathbb{M}$.

DEFINITION 1.29. Let $H \vdash I : \mathbb{C} \to \mathbb{M}$ be a semi-left-exact reflection from a finitely complete category $\mathbb{C}$ into a full subcategory $\mathbb{M}$, and $\sigma : S \to R$ a morphism of Galois descent.

The Galois groupoid $Gal[\sigma]$ of σ is the following internal groupoid in $\mathbb{M}$,

(1.34)
$$\begin{tikzcd}[column sep=large] I(S \times_R S) \times_{I(S)} I(S \times_R S) \arrow[r, shift left=4, "I(f_0)"] \arrow[r, "I(m)" description] \arrow[r, shift right=4, "I(f_1)"'] & I(S \times_R S) \arrow[r, shift left=4, "I(\pi_2)"] \arrow[r, shift right=4, "I(\pi_1)"'] & I(S) \arrow[l, "I(\Delta)" description] \end{tikzcd}$$

given by Lemma 1.28, with $I(\tau) : I(S \times_R R) \to I(S \times_R R)$ the twisting morphism, using the terminology of Proposition 1.13.

DEFINITION 1.30. An internal covariant presheaf on the internal groupoid $\mathbb{G}$ consists in the data in the following diagram, where $\mathbb{G}$ is its bottom line,

$$
\begin{tikzcd}
P_2 \arrow[r, shift left=2, "f_0'"] \arrow[r, "m'" description] \arrow[r, shift right=2, "f_1'"'] \arrow[d, "p_2"'] & P_1 \arrow[r, shift left=2, "d_0'"] \arrow[r, shift right=2, "d_1'"'] \arrow[d, "p_1"'] & P_0 \arrow[l, "n'" description] \arrow[d, "p_0"'] \\
C_2 \arrow[r, shift left=2, "f_0"] \arrow[r, "m" description] \arrow[r, shift right=2, "f_1"'] & G_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & G_0 \arrow[l, "n" description]
\end{tikzcd} \tag{1.35}
$$

where all the squares of morphisms, corresponding to each other by the notation, are pullbacks.

THEOREM 1.31. *Let $H \vdash I : \mathbb{C} \to \mathbb{M}$ be a semi-left-exact reflection of a category $\mathbb{C}$ with pullbacks into a full subcategory $\mathbb{M}$ and $\sigma : S \to R$ a Galois descent morphism. Under these conditions, there exists an equivalence of categories*

$$\mathbf{Split}_R(\sigma) \cong \mathbb{M}^{Gal[\sigma]},$$

between the category of those objects $(A,\ a) \in \mathbb{C}/R$ which are split by σ and the category of internal covariant presheaves on the internal groupoid $Gal[\sigma]$ in $\mathbb{M}$.

In fact, the composite functor $I_S \circ \sigma^*$ is monadic, so that $\mathbb{M}^{Gal[\sigma]}$ is equivalent to the category of algebras associated to the monadic functor $I_S \circ \sigma^*$.

If σ is just an effective descent morphism, there is still an equivalence $\mathbf{Split}_R(\sigma) \cong \mathbb{M}^{Gal[\sigma]}$, but now $Gal[\sigma]$ is not necessarily an internal groupoid, but what is called a pregroupoid (cf. [**1**] or [**2**]).

1.5. Pullbacks preservation in reflective subvarieties

Here we state necessary and sufficient conditions for the preservation of pullbacks, by the reflector I, into a subvariety of universal algebras, that will be used forward in this text.

DEFINITION 1.32. Let $\mathbb{A}$ be a category and $f : A \to B$, $g : A \to C$ be morphisms in $\mathbb{A}$. We say that f and g are jointly monic if

$$f \circ \alpha = f \circ \beta, \quad g \circ \alpha = g \circ \beta \quad \Rightarrow \quad \alpha = \beta,$$

for all $\alpha : X \to A$, $\beta : X \to A$ in $\mathbb{A}$.

(1) Let $\mathbb{A}$ be a finitely complete category, and let $\mathbb{B}$ be a full reflective subcategory of $\mathbb{A}$, the inclusion functor being $H : \mathbb{B} \to \mathbb{A}$, with unit $\eta : \mathbf{1}_{\mathbb{A}} \to HI$.

(2) Let $U : \mathbb{A} \to \mathbf{Set}$ be a functor that satisfies the following conditions (a), (b):

(a) $U(\eta_A)$ is surjective, for all $A \in \mathbb{A}$

(b) U preserves finite limits.

This is the case of varieties of universal algebras, since the unit morphisms of a reflection of a variety into a subvariety are surjective homomorphisms, and the underlying functor U from a variety into the category of sets has a left-adjoint, the free functor, hence U preserves finite limits.

Consider the following pullback diagram in $\mathbb{A}$

$$\begin{tikzcd} A \times_B C \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & C \arrow[d, "f"] \\ A \arrow[r, "g"] & B \end{tikzcd} \tag{1.36}$$

and the following pullback in $\mathbb{B}$

$$
\begin{tikzcd}
HI(A \times_B C) \arrow[dr, "w"] \arrow[drr, "HI(\pi_2)"] \arrow[ddr, "HI(\pi_1)"'] & & \\
& HI(A) \times_{HI(B)} HI(C) \arrow[r, "p_2"] \arrow[d, "p_1"] & HI(C) \arrow[d, "HI(f)"] \\
& HI(A) \arrow[r, "HI(g)"] & HI(B)
\end{tikzcd}
\tag{1.37}
$$

The morphisms $HI(\pi_1)$, $HI(\pi_2)$ are jointly monic if and only if w is a monomorphism. Hence $U(w)$ is an injective map, since U preserves finite limits, and the following conditions (1) and (2) are equivalent:

(1) $U(w)$ is an injection.

(2) If $U(\eta_A)(a) = U(\eta_A)(a')$ and $U(\eta_C)(c) = U(\eta_C)(c')$, then $U(\eta_{A\times_B C})(a, c) = U(\eta_{A\times_B C})(a', c')$, for every (a, c), $(a', c') \in U(A \times_B C)$.

(1) $\Leftrightarrow$ (2):

(2) $\Rightarrow$ (1):
Let α, $\beta \in UHI(A \times_B C)$ be such that $U(w)(\alpha) = U(w)(\beta)$. Since $U(\eta_{A\times_B C})$ is a surjective map, there exist (a, c), $(a', c') \in U(A \times_B C)$, such that $U(\eta_{A\times_B C})(a, c) = \alpha$, $U(\eta_{A\times_B C})(a', c') = \beta$. Hence, $U(\eta_A)(a) = U(\eta_A)(a')$ and $U(\eta_C)(c) = U(\eta_C)(c')$. Therefore, $U(\eta_{A\times_B C})(a, c) = U(\eta_{A\times_B C})(a', c')$, that is, $\alpha = \beta$.

(1) $\Rightarrow$ (2):
Conversely, suppose that $U(w)$ is an injective map and $U(\eta_A)(a) = U(\eta_A)(a')$, $U(\eta_C)(c) = U(\eta_C)(c')$, for (a, c), $(a', c') \in U(A\times_B C)$. Then,

$$U(p_2) \circ U(w) \circ U(\eta_{A\times_B C})(a, c) = U(p_2) \circ U(w) \circ U(\eta_{A\times_B C})(a', c'),$$

$U(p_1) \circ U(w) \circ U(\eta_{A\times_B C})(a,c) = U(p_1) \circ U(w) \circ U(\eta_{A\times_B C})(a',c')$.

Hence, $U(w) \circ U(\eta_{A\times_B C})(a,c) = U(w) \circ U(\eta_{A\times_B C})(a',c')$.

Thus $U(\eta_{A\times_B C})(a,c) = U(\eta_{A\times_B C})(a',c')$, since $U(w)$ is an injection.

If the monomorphisms are just the injective morphisms, in the category $\mathbb{A}$, that is, if U reflects monomorphisms, then $HI(\pi_1)$, $HI(\pi_2)$ are jointly monic if and only if the equivalent conditions (1), (2) hold.

That is the case, for instance, of any variety $\mathbb{C}$ of universal algebras: Since the underlying functor $U : \mathbb{C} \to \mathbf{Set}$ preserves finite limits,the image by U of a monomorphism is an injective map; and, on the other hand, every injective homomorphism is a monomorphism.

Thus, in this case, $HI(\pi_1)$ and $HI(\pi_2)$ are jointly monic if and only if $a \sim_A a^*$ and $c \sim_C c^*$ imply $(a,c) \sim_{A\times_B C} (a^*,c^*)$, for all $(a,c),\ (a^*,c^*) \in A \times_B C$.

Notice that the product $A \times B$ in a finitely complete category $\mathbb{A}$ is the following pullback, where T is the terminal object in $\mathbb{A}$.

(1.38)
$$\begin{tikzcd}
A \times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "!"] \\
A \arrow[r, "!"] & T
\end{tikzcd}$$

Consider, again, the pullbacks (1.36) and (1.37).

The following conditions (3) and (4) are equivalent:

(3) $U(w)$ is a surjection.

(4) For all $c \in U(C),\ a \in U(A)$ such that $U(\eta_B \circ f)(c) = U(\eta_B \circ g)(a)$, there exists $(a',c') \in U(A \times_B C)$, such that $U(\eta_C)(c') = U(\eta_C)(c),\ U(\eta_A)(a') = U(\eta_A)(a)$.

(3) $\Leftrightarrow$ (4):

Suppose that $U(w)$ is a surjection and let $a \in U(A),\ c \in U(C)$ be such that $U(\eta_B \circ f)(c) = U(\eta_B \circ g)(a)$. Then, $U(HI(f) \circ \eta_C)(c) =$

$U(HI(g) \circ \eta_A)(a)$. Hence, $(U(\eta_A)(a), U(\eta_C)(c)) \in U(HI(A) \times_{HI(B)} HI(A))$.

Since $U(w)$ and $U(\eta_{A\times_B C})$ are surjections, there exists $(a', c') \in U(A \times_B C)$, such that $U(w \circ \eta_{A\times_B C})(a', c') = (U(\eta_A)(a), U(\eta_C)(c))$, i.e., $U(HI(\pi_2) \circ \eta_{A\times_B C})(a', c') = U(\eta_C)(c)$ and $U(HI(\pi_1) \circ \eta_{A\times_B C})(a', c') = U(\eta_A)(a)$. Therefore, $U(\eta_C)(c') = U(\eta_C)(c)$ and $U(\eta_A)(a') = U(\eta_A)(a)$.

Conversely, consider $(\alpha, \beta) \in U(HI(A) \times_{HI(B)} HI(C))$. Then, $UHI(f)(\beta) = UHI(g)(\alpha)$. Since $U(\eta_C)$, $U(\eta_A)$ are surjections, there exist $c^* \in U(C)$, $a^* \in U(A)$, such that $U(\eta_C)(c^*) = \beta$, $U(\eta_A)(a^*) = \alpha$. Thus, $U(\eta_B \circ f)(c^*) = U(\eta_B \circ g)(a^*)$.

By hypothesis, there exist $(a', c') \in U(A \times_B C)$, such that $U(\eta_C)(c') = U(\eta_C)(c^*)$, $U(\eta_A)(a') = U(\eta_A)(a^*)$. Hence, $U(HI(\pi_2) \circ \eta_{A\times_B C})(a', c') = U(\eta_C)(c') = \beta$ and $U(HI(\pi_1) \circ \eta_{A\times_B C})(a', c') = U(\eta_A)(a') = \alpha$.

Therefore, there exists $U(\eta_{A\times_B C})(a', c') \in UHI(A \times_B C)$ such that $U(w)(U(\eta_{A\times_B} C)(a', c')) = (\alpha, \beta)$.

In the case of varieties of universal algebras w is a surjective homomorphism if and only if, for every $a \in A$ and $c \in C$, such that $g(a) \sim_B f(c)$, there exist $(a^*, c^*) \in A \times_B C$, with $a^* \sim_A a$ and $c^* \sim_C c$.

CHAPTER 2

Preservation of finite products

In section 2.1 we will give a necessary and sufficient condition for the preservation of finite products by a reflector $I : \mathbb{C} \to \mathbb{M}$, from a variety of universal algebras $\mathbb{C}$ into an idempotent subvariety $\mathbb{M}$, namely, $I(F(x) \times F(x)) = \mathbf{1}$. If the variety $\mathbb{C}$ is idempotent then the reflector I preserves finite products as a consequence of this result, since $F(x) = \mathbf{1}$. This is generalized, in section 2.2, for reflections of categories $H \vdash I : \mathbb{C} \to \mathbb{M}$ which have a functor $U : \mathbb{C} \to \mathbf{Set}$ such that U reflects isomorphisms, U preserves finite limits and $U(\eta_C) : U(C) \to U(HI(C))$ is a surjection, for every unit morphism.

Under the conditions above we conclude, by Lemmas 1.10 and 1.11, that those reflections have stable units if and only if they are semi-left-exact.

2.1. Varieties into idempotent subvarieties

Consider the following data (2.1):

(1) A reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$, from a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$, with unit $\eta : 1_{\mathbb{C}} \to HI$.

(2) The adjunction $(F, U, \lambda, \varepsilon) : \mathbf{Set} \to \mathbb{C}$, where F is the free functor and U is the underlying functor. $F(x)$ will denote the free algebra generated by the one-point set $\{x\}$, and $\mathbf{1}$ will denote the trivial algebra in $\mathbb{C}$ (which is a terminal object in $\mathbb{C}$).

Notice that for every C in $\mathbb{C}$, $\eta_C : C \to HI(C)$ is a surjective homomorphism, with $HI(C) = C/\sim_C$, where $\sim_C$ denotes the congruence on C associated to the reflection. We will use the notation $c \sim_C c'$, meaning $\eta_C(c) = \eta_C(c')$, for $c,\ c' \in C$.

Lemma 2.1. *If $I(F(x) \times F(x)) = \mathbf{1}$ then the reflector I preserves finite products.*

Proof. Let Q and R be objects in $\mathbb{C}$. The map

$$\begin{aligned} q : \{x\} &\to U(Q) \\ x &\mapsto q \end{aligned}$$

extends uniquely to a homomorphism $h_q : F(x) \to Q$, because $\lambda_{\{x\}} : \{x\} \to UF(x)$ is universal from $\{x\}$ to U.

For any $(q, r) \in Q \times R$, there exists a unique homomorphism $h_q \times h_r$ which makes the following product diagram commute

$$\begin{tikzcd} Q & Q \times R \arrow[l, "\pi_Q"'] \arrow[r, "\pi_R"] & R \\ F(x) \arrow[u, "h_q"'] & F(x) \times F(x) \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "h_q \times h_r"'] & F(x) \arrow[u, "h_r"'] \end{tikzcd} \tag{2.2}$$

Since $\eta : 1_{\mathbb{C}} \to HI$ is a natural transformation, the following diagram commutes

$$\begin{tikzcd} F(x) \times F(x) \arrow[r, "\eta_{F(x)\times F(x)}"] \arrow[d, "h_q \times h_r"] & HI(F(x) \times F(x)) \arrow[d, "HI(h_q \times h_r)"] \\ Q \times R \arrow[r, "\eta_{Q\times R}"] & HI(Q \times R) \end{tikzcd} \tag{2.3}$$

Since $I(F(x) \times F(x)) = \mathbf{1}$,

$$(h_q(w_1), h_r(w_2)) \sim_{Q\times R} (h_q(w_3), h_r(w_4)), \tag{2.4}$$

for all $q \in Q$, $r \in R$ and for all $w_1, w_2, w_3, w_4 \in F(x)$, where $\sim_{Q\times R}$ is the congruence associated to the surjective homomorphism $\eta_{Q\times R} : Q \times R \to HI(Q \times R)$.

Fixing $r \in R$, the following map

$$\begin{aligned} \varphi : Q &\to HI(Q \times R) \\ q &\mapsto [(q,r)]_{\sim_{Q\times R}} \end{aligned}$$

is a homomorphism:

Let $q_1, ..., q_n \in Q$ and let θ be an operator on $\mathbb{C}$, of arity $n \in \mathbf{N_0}$. Since $\theta_Q(q_1, ..., q_n) = q = h_q(x)$, for some $q \in Q$ and $r = h_r(x)$, then,

$$\varphi(\theta_Q(q_1, ..., q_n)) = [(h_q(x), h_r(x))]_{\sim_{Q\times R}} =$$

$= [(h_q(x), h_r(\theta_{F(x)}(x, ..., x)))]_{\sim_{Q\times R}}$, by (2.4) above,

$= [(\theta_Q(q_1, ..., q_n), \theta_R(r, ..., r))]_{\sim_{Q\times R}}$, because h_r is a homomorphism,

$= [\theta_{Q\times R}((q_1, r), ..., (q_n, r))]_{\sim_{Q\times R}}$, by definition of $\theta_{Q\times R}$ on the product of universal algebras,

$= \theta_{HI(Q\times R)}([(q_1, r)]_{\sim_{Q\times R}}, ..., [(q_n, r)]_{\sim_{Q\times R}})$, since $\eta_{Q\times R} : Q \times R \to HI(Q \times R)$ is a homomorphism,

$= \theta_{HI(Q\times R)}(\varphi(q_1), ..., \varphi(q_n))$.

Since $\eta_Q : Q \to HI(Q)$ is universal from Q to H, $\varphi : Q \to HI(Q \times R)$ induces a homomorphism

$$\begin{aligned} h : I(Q) &\to I(Q \times R) \\ [q]_Q &\mapsto [(q,r)]_{\sim_{Q\times R}}. \end{aligned}$$

One can construct,by analogous arguments, a homomorphism

$$\begin{aligned} h' : I(R) &\to I(Q \times R) \\ [r]_Q &\mapsto [(q,r)]_{\sim_{Q\times R}}, \end{aligned}$$

for every fixed $q \in Q$.

Then, $q \sim_Q q^*$ implies $(q, r) \sim_{Q\times R} (q^*, r)$ and $r \sim_R r^*$ implies $(q^*, r) \sim_{Q\times R} (q^*, r^*)$. Therefore,

$$q \sim_Q q^* \text{ and } r \sim_R r^* \text{ implies } (q, r) \sim_{Q\times R} (q^*, r^*). \tag{2.5}$$

Consider the diagram:

$$
\begin{tikzcd}
HI(Q) & & HI(Q)\times HI(R) \arrow[ll, "p_1"'] \arrow[rr, "p_2"] & & HI(R) \\
& & HI(Q\times R) \arrow[ull, "HI(\pi_Q)"] \arrow[u] \arrow[urr, "HI(\pi_R)"'] & & \\
& Q \arrow[uul, "\eta_Q"] & Q\times R \arrow[l, "\pi_Q"'] \arrow[r, "\pi_R"] \arrow[u, "\eta_{Q\times R}"'] & R \arrow[uur, "\eta_R"'] &
\end{tikzcd}
\tag{2.6}
$$

Since η_R and η_Q are surjective homomorphisms, $\langle HI(\pi_Q), HI(\pi_R)\rangle \circ \eta_{Q\times R}$ is also a surjective homomorphism. Hence, $\langle HI(\pi_Q), HI(\pi_R)\rangle$ is a surjective homomorphism. On the other hand, $HI(\pi_Q)$ and $HI(\pi_R)$ are jointly monic, by (2.5). Hence, $\langle HI(\pi_Q), HI(\pi_R)\rangle$ is a bijective homomorphism. Therefore, $I(Q\times R) \cong I(Q)\times I(R)$. □

EXAMPLE 2.2. The reflection of **SGr** (variety of semigroups) into **Band** (variety of idempotent semigroups) (see [**9**]).

COROLLARY 2.3. *If every element in any $C \in \mathbb{C}$ is a subalgebra then the reflector I preserves finite products.*

PROOF. Since every element in any $C \in \mathbb{C}$ is a subalgebra (which implies that every element in any $M \in \mathbb{M}$ is a subalgebra) every element x in any $C \in \mathbb{C}$ satisfies $x = \theta(x, ..., x)$, for every n-ary operation θ on C, with $n \in \mathbf{N_0}$. Hence, $F(x) = \mathbf{1}$ and $F(x)\times F(x) = \mathbf{1}$. Therefore, $I(F(x)\times F(x)) = I(\mathbf{1})$. Since $\eta_{\mathbf{1}} : \mathbf{1} \to HI(\mathbf{1})$ is an isomorphism, $I(F(x)\times F(x)) = \mathbf{1}$. Hence, by Lemma 2.1, I preserves finite products. □

EXAMPLE 2.4. The reflection of **Band** into **SLat** (variety of semilattices).

COROLLARY 2.5. *If* $H \vdash I : \mathbb{C} \to \mathbb{M}$ *is a reflection from a variety of universal algebras into an idempotent subvariety, then the following conditions are equivalent:*

(1) *I preserves finite products;*

(2) *I preserves the product* $F(x) \times F(x)$;

(3) $I(F(x) \times F(x)) = \mathbf{1}$.

PROOF. If I preserves finite products, then I preserves the product $F(x) \times F(x)$.

If I preserves the product $F(x) \times F(x)$, that is, $I(F(x) \times F(x)) = I(F(x)) \times I(F(x))$, then $I(F(x) \times F(x)) = \mathbf{1}$, since $\mathbb{M}$ is idempotent.

If $I(F(x) \times F(x)) = \mathbf{1}$, then I preserves finite products, by Lemma 2.1.

□

Proposition 2.6 below, follows from Lemmas 2.1, 1.11 and 1.10.

PROPOSITION 2.6. *For a reflection* $H \vdash I : \mathbb{C} \to \mathbb{M}$, *from a variety of universal algebras* $\mathbb{C}$ *into a subvariety* $\mathbb{M}$, *the following conditions are equivalent:*

(a) *$H \vdash I$ has stable units and every element in any $X \in \mathbb{M}$ is a subalgebra;*

(b) *$H \vdash I$ has stable units and* $I(F(x)) = \mathbf{1}$;

(c) *$H \vdash I$ has stable units and* $I(F(x) \times F(x)) = \mathbf{1}$;

(d) *$H \vdash I$ is semi-left exact and* $I(F(x) \times F(x)) = \mathbf{1}$;

(e) *If C is either a connected component(cf. Definition 1.9) or $C = F(x) \times F(x)$ then* $I(C) = \mathbf{1}$.

PROOF. First notice that the condition "every element in any $X \in \mathbb{M}$ is a subalgebra" is equivalent to $U_{\mathbf{1},M} : \mathbb{C}(\mathbf{1}, M) \to \mathbf{Set}(\{*\}, U(M))$ is a surjection, for every $M \in \mathbb{M}$.

(a)$\Leftrightarrow$(b):

Every element x in any $X \in \mathbb{M}$ is a subalgebra, if and only if every $x \in X$ satisfies $x = \theta(x, ..., x)$ for every n-ary ($n \in \mathbf{N_0}$) operation on any $X \in \mathbb{M}$, if and only if $HI(F(x)) = \mathbf{1}$, because $HI(F(x))$ is the free algebra generated by $\{x\}$ in the subvariety $\mathbb{M}$, where $U(\eta_{F(x)}) \circ \lambda_{\{x\}}$ is the universal map from $\{x\}$ to U.

(b)$\Leftrightarrow$(c):

Consider the following commutative square

$$\begin{tikzcd} F(x) \times F(x) \arrow[r, "\pi"] \arrow[d, "\eta_{F(x)\times F(x)}"] & F(x) \arrow[d, "\eta_{F(x)}"] \\ HI(F(x) \times F(x)) \arrow[r, "HI(\pi)"] & HI(F(x)) \end{tikzcd} , \tag{2.7}$$

where π is a product projection.

Since π and $\eta_{F(x)}$ are surjective homomorphisms, $HI(\pi)$ is a surjective homomorphism. If $HI(F(x) \times F(x)) = \mathbf{1}$ then $HI(\pi)$ is an injective homomorphism. Hence, $HI(F(x)) = \mathbf{1}$.

Conversely, by Lemma 1.11, if $H \vdash I$ has stable units then $I(C_\mu \times D_\nu) = \mathbf{1}$, with C_μ, D_ν connected components. Consider the pullback diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathbf{1} \arrow[d] \\ F(x) \arrow[r, "\eta_{F(x)}"] & HI(F(x)) \end{tikzcd} , \tag{2.8}$$

where $HI(F(x)) = \mathbf{1}$. Then, $F(x) \cong P$. Thus, $F(x)$ is a connected component. Therefore, $I(F(x) \times F(x)) = \mathbf{1}$.

(c)$\Leftrightarrow$(d):

If $H \vdash I$ is semi-left exact then, by Lemma 1.10, $I(C_\mu) = \mathbf{1}$ for every connected component C_μ. Since $I(F(x) \times F(x)) = \mathbf{1}$, by Lemma 2.1, I preserves finite products. Hence, $I(C_\mu \times D_\nu) = I(C_\mu) \times I(D_\nu) = \mathbf{1}$, for every pair of connected components C_μ, D_ν. Therefore, by Lemma 1.11, $H \vdash I$ has stable units. The converse is always true.

(d)$\Leftrightarrow$(e):

$H \vdash I$ is semi-left exact if and only if $I(C_\mu) = \mathbf{1}$ for every connected component C_μ, by Lemma 1.10.

$\square$

Considering Lemmas 1.10, 1.11 and Corollary 2.3 it is straightforward to conclude:

PROPOSITION 2.7. *If every element in any $C \in \mathbb{C}$ is a subalgebra then, $I \dashv H$ is semi-left-exact if and only if $I \dashv H$ has stable units.*

From Examples 2.2 and 2.4 one can conclude that the reflection $H \vdash I : \mathbf{SGr} \to \mathbf{SLat}$ has stable units if and only if it is semi-left-exact. In [**9**] is proved that this reflection has stable units, while in [**12**] is proved that this reflection is semi-left-exact.

Notice that $I(F(x)) = \mathbf{1}$ does not imply $I(F(x) \times F(x)) = \mathbf{1}$, as in the following Example 2.8:

EXAMPLE 2.8. Let M be a monoid, and $X \in \mathbf{Set}$. The universal algebra with unary operations $m(x) = mx$, $1(x) = x$ such that $m'(m(x)) = m'mx$ for every $m, m' \in M$, and $x \in X$ is an M-**Set**.

Every $X \in \mathbf{Set}$ is an M-**Set**, S, since if we state $mx = x$, for all $m \in M$, $x \in X$ then $S = X$.

Consider the reflection of M-**Set** into **Set**, associated to the congruence generated, on every $S \in M$-**Set**, by $ms = s$, for all $m \in M$, and all $s \in S$.

A congruence contains $R = \{(s, ms) \mid s \in S; m \in M\}$ if and only if it contains $R^* = \{(ms, m's) \mid s \in S; m, m' \in M\}$, by transitivity. Therefore, R and R^* generate the same congruence.

Let C_S be the following subset of $S \times S$:

(2.9)
$$\{(a, b) \in S{\times}S \mid (\exists\, z_0, .., z_n \in S; n \in \mathbf{N})\, a = z_0,\ b = z_n,\ (z_i, z_{i+1}) \in R^*; i = 0, .., n{-}1\}$$

Since C_S is the transitive closure of R^*, C_S is contained in every congruence that contains R^*.

C_S is the congruence generated by R, because:

(i) $\{(a, a) \mid a \in S\} \subseteq C_S$;

(ii) C_S is obviously symmetric and transitive;

(iii) C_S respects the (unary) operations on S, since if there exists a finite sequence as in (2.9) between a and b then, there exists a finite sequence as in (2.9) between ma and mb, for all $m \in M$.

In M-**Set**, $F(x) = M$, $F(x){\times}F(x) = M{\times}M$ and, clearly, $I(F(x)) = \mathbf{1}$.

On the other hand, $I(F(x){\times}F(x)) = \mathbf{1}$ if and only if $(1_M, 1_M) \sim_{M\times M} (m, m')$, for all $m, m' \in M$.
According to (2.9), $(1_M, 1_M) \sim_{M\times M} (m, m')$ if and only if there exists a finite sequence:

(2.10)
$$(1_M, 1_M) = (m_0, m_0'), (m_1, m_1'), ..., (m_i, m_i'), (m_{i+1}, m_{i+1}'), ..., (m_n, m_n') = (m, m')$$

such that, for every pair $((m_i, m_i'), (m_{i+1}, m_{i+1}'))$, $(m_i, m_i') = c(a, b)$, $(m_{i+1}, m_{i+1}') = d(a, b)$, for some $a, b, c, d \in M$, i.e., there exist $a, b, c, d \in M$, that satisfy $ca = m_i$, $cb = m_i'$, $da = m_{i+1}$, $db = m_{i+1}'$, $i = 0, 1, ..., n{-}1$, with $n \in \mathbf{N}$.

If $M \neq \{1_M\}$ is a monoid with left-cancellation law then, $[(1_M, 1_M)]_{\sim_{M\times M}} \neq [(m, m')]_{\sim_{M\times M}}$ for $m \neq m'$, with $m, m' \in M$, as can be easily checked by induction on the length of the finite sequence (2.10):

Let $n = 1$. If $(1_M, 1_M) \sim_{M\times M} (m, m')$, then there exist $a,\ b,\ c,\ d \in M$ which satisfy $ca = 1_M,\ cb = 1_M,\ da = m,\ db = m'$. Since M has left-cancellation law $ca = cb \Rightarrow a = b$. Hence, $m = m'$.

Suppose that for any sequence as in (2.9) of length n, between $(1_M, 1_M)$ and (m, m'), for every pair $((m_i, m'_i), (m_{i+1}, m'_{i+1}))$ we have $m_i = m'_i$, $m_{i+1} = m'_{i+1}$.

Then, for a sequence as in (2.9) of length $n+1$, between $(1_M, 1_M)$ and (m, m'), we have $(1_M, 1_M)$, (m_1, m_1), ...
..., (m_{n-1}, m_{n-1}), (m, m'), such that there exist a, b, c, $d \in M$ with $ca = m_{n-1}$, $cb = m_{n-1}$, $da = m$, $db = m'$. By left-cancellation law, $a = b$. Hence $m = m'$. Therefore, $I(F(x) \times F(x)) \neq \mathbf{1}$.

REMARK 2.9. It is well known that this reflection, of M-**Set** into **Set**, is semi-left-exact, which follows from more general results(see [**1**]). Here we show it, for this particular case, using Lemma 1.10. The reflection $H \vdash I : M\text{-}\mathbf{Set} \to \mathbf{Set}$ is semi-left-exact if and only if every connected component is connected, by Lemma 1.10.

First notice that since $HIF(x) = \mathbf{1}$, each class $[a]_{C_S} = \{b \in S \mid (a,b) \in C_S\}$ is a subalgebra of S.

A connected component is connected if for each class $[a]_{C_S} = \{b \in S \mid (a,b) \in C_S\}$, $HI([a]_{C_S}) = \mathbf{1}$, i.e., if $(a,b) \in C_S$ then, there exists a sequence as in (2.9), where $R^* = \{(ms, m's) \mid s \in [a]_{C_S}; m, m' \in M\}$.

Suppose that $(a,b) \in C_S$ then, there exists a finite sequence $a = z_0$, z_1, ..., $z_n = b$, such that $(z_i, z_{i+1}) = (m_i s_i, m'_i s_i)$, for $s_i \in S$, and $m_i, m'_i \in M$ $i = 0, 1, ..., n-1$. We want to prove that $s_i \in [a]_{C_S}$, for every $i = 0, 1, ..., n-1$.

If $n = 1$ then $a = ms$, $b = m's$. Since $(s, ms) \in C_S$, $s \in [a]_{C_S}$.

Suppose that for all sequences of length $n \in \mathbf{N}$:
$a = z_0 = m_0 s_0$, $z_1 = m_1 s_0 = m'_1 s_1$, $z_2 = m_2 s_1 = m'_2 s_2, ...$
..., $z_{n-2} = m_{n-2} s_{n-3} = m'_{n-2} s_{n-2}$, $z_{n-1} = m_{n-1} s_{n-2}$,
with $m_i, m'_i \in M$, $s_i \in S$, $i = 0, 1, ..., n-1$, we have $s_i \in [a]_{C_S}$, $i = 0, 1, ..., n-2$.

Consider a sequence of length $n+1$, $a = z_0 = m_0 s_0$, $z_1 = m_1 s_0 = m'_1 s_1$, $z_2 = m_2 s_1 = m'_2 s_2, ..., z_{n-1} = m_{n-1} s_{n-2} = m'_{n-1} s_{n-1}$, $z_n = m_n s_{n-1}$, with $m_i, m'_i \in M$, $s_i \in S$, $i = 0, 1, ..., n$.

By hypothesis, $s_0, ..., s_{n-2} \in [a]_{\sim C_S}$. On the other hand, since $(m_n s_{n-1}, s_{n-1}) \in C_S$, $(z_n, s_{n-1}) \in C_S$. Since $(a, z_n) \in C_S$, $s_{n-1} \in [a]_{C_S}$.

On the other hand, if M is a monoid with a left-zero, 0_M, $I(F(x) \times F(x)) = \mathbf{1}$, because for all $m,\ m' \in M$, there exist $a,\ b,\ c,\ d \in M$ which satisfy $ca = 0_M,\ cb = 0_M,\ da = m,\ db = m'$, namely, $c = 0_M;\ a = m;\ b = m';\ d = 1_M$. Therefore, $(0_M, 0_M) \sim_{M \times M} (m, m')$.

Hence, $H \vdash I : M\text{-}\mathbf{Set} \to \mathbf{Set}$ has stable units, if M has a left-zero, since I preserves finite products, by Lemma 2.1 and the reflection is semi-left-exact.

REMARK 2.10. The reflection $H \vdash I : M\text{-}\mathbf{Set} \to \mathbf{Set}$ does not have stable units, provided M is a monoid with left-cancellation law.

By Lemma 1.11, $H \vdash I : M\text{-}\mathbf{Set} \to \mathbf{Set}$ has stable units if and only if $HI(C_\mu \times D_\nu) = \mathbf{1}$, where C_μ, D_ν are connected components associated to the morphisms μ, ν, respectively.

Since $HI(M) = \mathbf{1}$, M is a connected component associated to this isomorphism. On the other hand, since M has left-cancellation law, $HI(M \times M) \neq \mathbf{1}$.

2.2. A more general setting

The following Lemma is a known result in **Set**.

LEMMA 2.11. *Consider the following commutative diagram in* **Set**, *where* $\alpha,\ \beta,\ \delta$ *are surjections and the bottom line is a product diagram.*

(2.11)
$$\begin{tikzcd}
B & A \arrow[l, "f"'] \arrow[r, "g"] & C \\
Q \arrow[u, "\beta"] & Q \times R \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] \arrow[u, "\alpha"'] & R \arrow[u, "\delta"']
\end{tikzcd}$$

The following conditions are equivalent:

(a) For every $r \in R$ the map $\gamma_r : Q \to A;\ q \mapsto \alpha(q, r)$ (in the left-hand commutative diagram) factorizes through β and for every $q \in Q$

the map $\gamma_q : R \to A;\ r \mapsto \alpha(q,r)$ *(in the right-hand commutative diagram) factorizes through* δ,

$$
\begin{tikzcd}
Q \arrow[r, "\gamma_r"] \arrow[dr] & A \\
& Q \times R \arrow[u, "\alpha"']
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "\gamma_q"] \arrow[dr] & A \\
& Q \times R \arrow[u, "\alpha"']
\end{tikzcd}
\tag{2.12}
$$

$$q \mapsto (q,r) \qquad\qquad r \mapsto (q,r) \quad ;$$

(b) the maps f *and* g *are jointly monic.*

Proof. Let $\gamma_r = \lambda_r \circ \beta$ and $\gamma_q = \lambda_q \circ \delta$, for some maps $\lambda_r : B \to A$, $\lambda_q : C \to A$, for every $r \in R$, and $q \in Q$.

Let $f(a) = f(a')$ and $g(a) = g(a')$, where $a = \alpha(q,r)$, $a' = \alpha(q',r')$, with $(q,r),\ (q',r') \in Q \times R$.

Let $\overline{a} = \alpha(q,r')$.

$f(\overline{a}) = \beta(q) = f(a)$ and $g(\overline{a}) = \delta(r') = g(a')$.

Hence, $f(a) = f(\overline{a})$ and $g(a) = g(\overline{a})$, with $a = \alpha(q,r)$ and $\overline{a} = \alpha(q,r')$.

Since $g(a) = g(a')$, $\delta(r) = \delta(r')$.

Hence, $\alpha(q,r) = \gamma_q(r) = \lambda_q \circ \delta(r) = \lambda_q \circ \delta(r') = \gamma_q(r') = \alpha(q,r')$. Therefore $a = \overline{a}$.

On the other hand, $f(\overline{a}) = f(a')$ and $g(\overline{a}) = g(a')$, with $\overline{a} = \alpha(q,r')$ and $a' = \alpha(q',r')$.

Since $f(\overline{a}) = f(a')$, $\beta(q) = \beta(q')$.

Hence, $\alpha(q,r') = \gamma_{r'}(q) = \lambda_{r'} \circ \beta(q) = \lambda_r \circ \beta(q') = \gamma_{r'}(q') = \alpha(q',r')$. Therefore $a' = \overline{a}$.

Thus, $a = a'$.

Conversely, let f and g be jointly monic and consider the following diagram:

(2.13)
$$
\begin{tikzcd}
B \arrow[r, shift right, "\lambda_r"'] & A \arrow[l, shift right, "f"'] \arrow[r, "g"] & C \\
Q \arrow[u, "\beta"] \arrow[ur, "\gamma_r"] & Q \times R \arrow[l, "\pi_1"'] \arrow[u, "\alpha"'] \arrow[r, "\pi_2"] & R \arrow[u, "\delta"'] \\
& Q \arrow[ul, "id_Q"] \arrow[u] \arrow[ur] \arrow[ur, shift right=2, "r", "q"'] &
\end{tikzcd}
$$

It remains to prove that $\lambda_r : B \to A;\ \beta(q) \mapsto \alpha(q, r)$, is a map.

Let $\beta(q) = \beta(q')$. Then, $f(\alpha(q, r)) = \beta(q) = \beta(q') = f(\alpha(q', r))$.
On the other hand, $\gamma(q, r) = \gamma(q', r)$. Hence, $g(\alpha(q, r)) = g(\alpha(q', r))$.

Since g and f are jointly monic, $\alpha(q, r) = \alpha(q', r)$.

Therefore, $\gamma_r = \lambda_r \circ \beta$.

Using the obvious diagram we would conclude that $\gamma_q = \lambda_q \circ \delta$, with $\lambda_q : C \to A;\ \delta(r) \mapsto \alpha(q, r)$.

□

The following Proposition 2.12, which follows straightforward from Lemma 2.11 generalizes Lemma 2.1.

Assume the following data (2.14), which are the same as in (1.7) with the exception of (2)(d):

(1) A reflection $H \vdash I : \mathbb{A} \to \mathbb{B}$, of a category $\mathbb{A}$, with finite limits, into a full subcategory $\mathbb{B}$, with unit $\eta : 1_{\mathbb{A}} \to HI$;

(2) a functor $U : \mathbb{A} \to \mathbf{Set}$, such that:

(a) U preserves finite limits;

(b) UH reflects isomorphisms;

(c) every map $U(\eta_A) : U(A) \to UHI(A)$ is a surjection, for every unit morphism η_A, $A \in \mathbb{A}$.

Proposition 2.12. *Under data* (2.14) (1) *and* (2), *I preserves finite products provided for all $q \in U(Q)$ and $r \in U(R)$, where $Q, R \in \mathbb{A}$, there exist morphisms $\gamma_r : Q \to HI(Q \times R)$ and $\gamma_q : R \to HI(Q \times R)$, such that*

$$U(\gamma_r)(a) = U(\eta_{Q\times R})(a, r),$$

for all $a \in U(Q)$, with $r \in U(R)$ fixed.

$$U(\gamma_q)(b) = U(\eta_{Q\times R})(q, b),$$

for all $b \in U(R)$, with $q \in U(Q)$ fixed.

Proof. Since $\eta_Q : Q \to HI(Q)$ is universal from Q to H, it induces a morphism $\beta : I(Q) \to I(Q \times R)$, such that the following diagram commutes.

(2.15)
$$\begin{tikzcd}
Q \arrow[r, "\eta_Q"] \arrow[dr, "\gamma_r"'] & HI(Q) \arrow[d, "H(\beta)"] \\
& HI(Q \times R)
\end{tikzcd}$$

Applying the functor U to the diagram (2.15) we conclude that γ_r factorizes through the surjective map $U(\eta_Q)$.

By analogous arguments we can conclude that γ_q factorizes through the surjective map $U(\eta_R)$.

Consider the diagram:

$$
\begin{tikzcd}
UHI(Q) & & UHI(Q) \times UHI(R) \arrow[ll, "U(p_1)"'] \arrow[rr, "U(p_2)"] & & UHI(R) \\
& & UHI(Q \times R) \arrow[ull, "UHI(\pi_1)"] \arrow[u] \arrow[urr, "UHI(\pi_2)"'] & & \\
& U(Q) \arrow[uul, "U(\eta_Q)"] & U(Q \times R) \arrow[l, "U(\pi_1)"'] \arrow[r, "U(\pi_2)"] \arrow[u, "U(\eta_{Q\times R})"'] & U(R) \arrow[uur, "U(\eta_R)"'] &
\end{tikzcd}
\tag{2.16}
$$

By Lemma 2.11, $UHI(\pi_1)$, $UHI(\pi_2)$ are jointly monic. Hence, $\langle UHI(\pi_1), UHI(\pi_2)\rangle$ is injective.

On the other hand, since $U(\eta_Q)$ and $U(\eta_R)$ are surjective maps, $U(\eta_Q) \times U(\eta_R) = \langle UHI(\pi_1), UHI(\pi_2)\rangle \circ U(\eta_{Q\times R})$ is also a surjective map. Let $(\alpha, \beta) \in UHI(Q) \times UHI(R)$. Since $U(\eta_R)$, $U(\eta_Q)$ are surjections, there exist $r \in U(R)$, $q \in U(Q)$, such that $U(\eta_R)(r) = \beta$, $U(\eta_Q)(q) = \alpha$. Hence, $(q, r) \in U(Q \times R)$ is such that $U(\eta_R \circ \pi_2)(q, r) = \beta$, $U(\eta_Q \circ \pi_1)(q, r) = \alpha$.

Since $U(p_2) \circ (U(\eta_Q) \times U(\eta_R))(q, r) = U(\eta_R) \circ U(\pi_2)(q, r) = \beta$ and $U(p_1) \circ (U(\eta_Q) \times U(\eta_R))(q, r) = U(\eta_Q) \circ U(\pi_1)(q, r) = \alpha$, $(U(\eta_Q) \times U(\eta_R))(q, r) = (\alpha, \beta)$.

Since U reflects isomorphisms, $I(Q \times R) \cong I(Q) \times I(R)$.

□

Proposition 2.13. *Under data* (2.14), (1) *and* (2), *I preserves finite products, provided every map $U_{T;A} : \mathbb{A}(T, A) \to \mathbf{Set}(\{*\}, U(A))$, is a surjection, for every object $A \in \mathbb{A}$, with T a terminal object in $\mathbb{A}$.*

Proof. Let Q, R be objects of $\mathbb{A}$. For every $r \in U(R)$, consider f_r, the inclusion map of $\{r\}$ into $U(R)$. Since, by hypothesis, there exists a morphism $f : T \to R$, such that $U(f) = f_r$, and $\mathbb{A}$ has finite products, we have a morphism $id_Q \times f : Q \times T \to Q \times R$, as shows the diagram (2.17), such that $U(id_Q \times f) \cong id_{U(Q)} \times f_r : U(Q) \times \{r\} \to U(Q) \times U(R)$.

(2.17)
$$\begin{tikzcd} Q & Q \times R \arrow[l, "\pi_Q"'] \arrow[r, "\pi_R"] & R \\ Q \arrow[u, "id_Q"] & Q \times T \arrow[l, "\pi_Q"] \arrow[r, "\pi_T"'] \arrow[u, "id_Q \times f"] & T \arrow[u, "f"] \end{tikzcd}$$

Since T is a terminal object, there exists a unique morphism $! : Q \to T$, where $U(!) : U(Q) \to \{r\}$ is the unique map from $U(Q)$ to $\{r\}$ and then there exists a morphism $\langle id_Q, !\rangle : Q \to Q \times T$. Therefore, there exists a morphism
$\gamma_r := \eta_{Q\times R} \circ (id_Q \times f) \circ \langle id_Q, !\rangle : Q \to Q \times T \to Q \times R \to HI(Q \times R)$, such that:
$U(\gamma_r) = U(\eta_{Q\times R} \circ (id_Q \times f) \circ \langle id_Q, !\rangle) = U(\eta_{Q\times R}) \circ (id_{U(Q)} \times f_r) \circ \langle id_{U(Q)}, U(!)\rangle = U(\eta_{Q\times R}) \circ \langle id_{U(Q)},\ f_r \circ U(!)\rangle$.

Therefore, $U(\gamma_r)(q) = U(\eta_{Q\times R})(q, r)$, for all $q \in Q$, with $r \in U(R)$ fixed.

One can construct, for every $q \in U(Q)$, by analogous arguments, a morphism
$\gamma_q := \eta_{Q\times R} \circ (g \times id_R) \circ \langle !, id_R\rangle : R \to Q \times T \to Q \times R \to HI(Q \times R)$, such that:
$U(\gamma_q) = U(\eta_{Q\times R} \circ (g \times id_R) \circ \langle !, id_R\rangle) = U(\eta_{Q\times R}) \circ (f_q \times id_{U(R)}) \circ \langle U(!), id_{U(R)}\rangle = U(\eta_{Q\times R}) \circ \langle f_q \circ U(!),\ id_{U(R)}\rangle$, where $f_q : \{q\} \to U(Q)$ is the inclusion map.

Therefore, $U(\gamma_q)(r) = U(\eta_{Q\times R})(q, r)$, for all $r \in R$, with $q \in U(Q)$ fixed.

□

PROPOSITION 2.14. *Under data* (2.14), (1) *and* (2), *and under the conditions of Lemma* 2.12 *the reflection* $H \vdash I$ *has stable units if and only if is semi-left-exact.*

PROOF. The proof follows straightforward from Lemmas 1.10, 1.11 and 2.12.

□

Notice that, when the reflection is in the context of the universal algebras, the forgetful functor U into **Set** always reflects isomorphisms, since an isomorphism is just a bijective homomorphism. While in the context of topological categories the reflection of isomorphisms by the forgetful functor is the crucial point, in order to apply Propositions 2.13 and 2.14. On the other hand, for any point x in any topological space X there is a morphism $h_x : \mathbf{1} \to X$, which takes the only point in the terminal space $\mathbf{1}$ to $x \in X$. The following example is a well-known reflection from a category of topological spaces which satisfies Propositions 2.13 and 2.14.

EXAMPLE 2.15. Consider the reflection $H \vdash I : \mathbf{CompHaus} \to \mathbf{Stone}$ from the category of topological spaces which are compact and Hausdorff, into its full subcategory consisting of the spaces totally disconnected (in the sense that for any two distinct points there is a clopen which contains one of them but not the other).

$I(X)$ consists in the connected components of X equipped with the quotient topology. It is well-known that the forgetful functor from **CompHaus** into **Set** is monadic. Hence, the conditions of Propositions 2.13 and 2.14 hold. In fact, this reflection has stable units (cf.[**2**]).

CHAPTER 3

Vertical morphisms

In this chapter we show that a reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ of a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$ is simple if and only if it is semi-left-exact. We describe the factorization system $(\mathcal{E}_I, \mathcal{M}_I)$ derived from simple=semi-left-exact reflections of universal algebras.

We state necessary and sufficient conditions for the class $\mathcal{E}'_I$, of stably-vertical homomorphisms, to be $\mathcal{E}_I \cap \mathcal{F}$, for reflections into idempotent subvarieties, where $\mathcal{F} = \{e : X \to Y \in \mathbb{C} \mid \forall_{y\in Y}\ \langle y\rangle_Y \cap e(X) \neq \emptyset\}$. As examples we have the reflection of the variety of bands into the subvariety of semilattices and the reflection of the variety of commutative semigroups into the subvariety of semilattices.

We also generalize some of these results for reflections $H \vdash I : \mathbb{C} \to \mathbb{M}$ of a finitely complete category $\mathbb{C}$ into a full subcategory $\mathbb{M}$, such that there exists a functor $U : \mathbb{C} \to \mathbf{Set}$ which preserves finite limits and reflects isomorphisms.

3.1. Vertical morphisms and trivial coverings

In this section we will describe the factorization system $(\mathcal{E}_I, \mathcal{M}_I)$, of vertical and trivial covering morphisms, respectively, induced by a simple reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ from a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$.

The next Proposition 3.1 allows us to conclude, in particular, that a reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ of a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$ is simple if and only if it is semi-left-exact, since the unit morphism $\eta_C : C \to HI(C)$ is a surjective homomorphism, for every $C \in \mathbb{C}$. It is known that surjective homomorphisms are exactly the effective descent morphisms in a variety of universal algebras.

PROPOSITION 3.1. *Let $H \vdash I : \mathbb{A} \to \mathbb{B}$ be a reflection of a finitely complete category $\mathbb{A}$ into a full subcategory $\mathbb{B}$. If the unit morphism*

$\eta_C : C \to HI(C)$ is an effective descent morphism in $\mathbb{A}$, for all $C \in \mathbb{A}$, then $H \vdash I$ is simple if and only if $H \vdash I$ is semi-left-exact.

PROOF. $H \vdash I$ is semi-left-exact if and only if $\pi_2 \in \mathcal{E}_I$, in every pullback square of the following form, by Definition 1.6,

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & H(X) \arrow[d, "g"] \\ B \arrow[r, "\eta_B"] & HI(B) \end{tikzcd} \quad . \tag{3.1}$$

Consider the following commutative diagrams (3.2) and (3.3),

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & H(X) \arrow[r, "\eta_{H(X)}"] \arrow[d, "g"] \arrow[dr, phantom, "(1)"] & HIH(X) \cong H(X) \arrow[d, "HI(g) \cong g"] \\ B \arrow[r, "\eta_B"'] & HI(B) \arrow[r, "\eta_{HI(B)}"'] & HIHI(B) \cong HI(B) \end{tikzcd} \quad , \tag{3.2}$$

$$\begin{tikzcd} P \arrow[r, "\eta_P"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "(2)"] & HI(P) \arrow[r, "HI(\pi_2)"] \arrow[d, "HI(\pi_1)"] \arrow[dr, phantom, "(3)"] & HIH(X) \cong H(X) \arrow[d, "HI(g) \cong g"] \\ B \arrow[r, "\eta_B"'] & HI(B) \arrow[r, "HI(\eta_B)"'] & HIHI(B) \cong HI(B) \end{tikzcd} \quad . \tag{3.3}$$

First notice that $g, \pi_1 \in \mathcal{M}_I$, because $g \in \mathbb{B}$ (see §3 in [**2**]) and $\mathcal{M}_I$ is stable for pullbacks. Since the reflection is simple, the squares (1) and (2) are pullbacks. Hence, the outside square (3.3) is also a pullback. Since η_B is an effective descent morphism in $\mathbb{A}$, (3) is a pullback (cf. Lemma 4.6. in [**2**]. On the other hand, $HI(\eta_B)$ is an isomorphism, because $H \vdash I : \mathbb{A} \to \mathbb{B}$ is a reflection into a full subcategory. Hence, $HI(\pi_2)$ is also an isomorphism. Therefore, $\pi_2 \in \mathcal{E}_I$.

□

REMARK 3.2.

(1) Under the conditions of Lemma 2.1, the reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ has stable units if and only if it is simple, by Propositions 2.6 and 3.1.

(2) Under data (2.14) and under the conditions of Proposition 2.12, if, in addition, every unit morphism $\eta_A : A \to HI(A)$ is an effective descent morphism then, the reflection $H \vdash I : \mathbb{A} \to \mathbb{B}$ has stable units if and only if it is simple, by Propositions 2.14 and 3.1.

Consider a simple reflection, or equivalently, by Proposition 3.1, a semi-left-exact reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ from a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$.

The class $\mathcal{E}_I$, of vertical morphisms in $\mathbb{C}$, and the class $\mathcal{M}_I$, of trivial coverings, which constitute the factorization system $(\mathcal{E}_I, \mathcal{M}_I)$ induced by the simple reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ are given in the next two Propositions 3.3 and 3.5.

PROPOSITION 3.3. *A homomorphism $f : A \to B$ in $\mathbb{C}$ belongs to $\mathcal{E}_I$ if and only if the following two conditions hold:*

(1) *for each $b \in B$, $[b]_{\sim_B} \cap f(A) \neq \emptyset$;*

(2) *for all $a, a^* \in A$, if $f(a) \sim_B f(a^*)$ then, $a \sim_A a^*$.*

PROOF. Consider the following commutative square:

(3.4)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \\ HI(A) \arrow[r, "HI(f)"] & HI(B) \end{tikzcd}$$

Suppose that $HI(f)$ is an isomorphism. Let $[b]_{\sim_B}$ be an arbitrary congruence class in B. Then, $\eta_B([b]_{\sim_B}) = \{\beta\}$, for some $\beta \in HI(B)$. Since $HI(f)$ and η_A are surjective homomorphisms, there exists $a \in A$, such that $HI(f)\circ\eta_A(a) = \beta$. Hence, $\eta_B(f(a)) = \beta$, that is, $f(a) \in [b]_{\sim_B}$. Therefore, $[b]_{\sim_B} \cap f(A) \neq \emptyset$.

Let $f(a) \sim_B f(a^*)$, that is, $\eta_B(f(a)) = \eta_B(f(a^*))$. Then, $HI(f) \circ \eta_A(a) = HI(f) \circ \eta_A(a^*)$. Since $HI(f)$ is a monomorphism, $\eta_A(a) = \eta_A(a^*)$, that is, $a \sim_A a^*$.

Conversely, let $\beta \in HI(B)$. Since η_B is a surjective homomorphism, there exists $b \in B$, such that $\eta_B(b) = \beta$. Since $[b]_{\sim_B} \cap f(A) \neq \emptyset$, there exists $a \in A$, such that $\eta_B \circ f(a) = \beta$. Hence, $HI(f)(\eta_A) = \beta$. Thus, $HI(f)$ is a surjective homomorphism.

Let $HI(f)(\alpha) = HI(f)(\gamma)$, with $\alpha,\ \gamma \in HI(A)$. Since η_A is a surjective homomorphism, there exist $a,\ a^* \in A$, such that $\eta_A(a) = \alpha,\ \eta_A(a^*) = \gamma$. Hence, $\eta_B(f(a)) = \eta_B(f(a^*))$, that is, $f(a) \sim_B f(a^*)$. Thus, $a \sim_A a^*$, that is, $\alpha = \eta_A(a) = \eta_A(a^*) = \gamma$.

$\square$

EXAMPLE 3.4. Consider the reflection $H \vdash I : \mathbf{Band} \to \mathbf{SLat}$ and let S be a band as in the following multiplication table.

(3.5)

$\bullet$	a	b
a	a	a
b	b	b

The homomorphism $e : S \to S$, given by $e(a) = e(b) = a$ belongs to $\mathcal{E}_I$.

Notice that $a = aba$ and $b = bab$, thus $a \sim_S b$. Hence, $HI(S) = \mathbf{1}$. Therefore, $HI(e)$ is a bijection, that is, an isomorphism.

Consider the reflection $H \vdash I : \mathbf{CommSgr} \to \mathbf{SLat}$ and let $\mathbf{N}$ be the commutative semigroup of the positive integers and let $\mathbf{R}^+$ be the commutative semigroup of the positive real numbers. Since both $\mathbf{N}$ and $\mathbf{R}^+$ are archimedean, $HI(\mathbf{N}) = HI(\mathbf{R}^+) = \mathbf{1}$. Hence, the inclusion homomorphism $\subseteq : \mathbf{N} \to \mathbf{R}^+$ belongs to $\mathcal{E}_I$.

In general, given a reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ of a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$, if $HI(A) = HI(B) = \mathbf{1}$, with $A,\ B \in \mathbb{C}$ then, any homomorphism $e : A \to B$, in $\mathbb{C}$, belongs to $\mathcal{E}_I$.

Proposition 3.5. *A homomorphism $f : A \to B$ in $\mathbb{C}$ belongs to $\mathcal{M}_I$ if and only if the following two conditions hold:*

(1) *for each $b \in B$, and for each $a \in A$, such that $b \sim_B f(a)$, there exists $c \in A$, for which $f(c) = b$ and $c \sim_A a$;*

(2) *for all $a, a^* \in A$, if $f(a) = f(a^*)$ and $a \sim_A a^*$ then $a = a^*$.*

In other words, f belongs to $\mathcal{M}_I$ if and only if its restrictions to the congruence classes $f_{|_{[a]\sim_A}} : [a]_{\sim_A} \to [f(a)]_{\sim_B}$ are bijections, for every $a \in A$.

Proof. Let $f : A \to B$ be a homomorphism in $\mathbb{C}$ and consider the following commutative square.

(3.6)
$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & HI(A) \arrow[d, "HI(f)"] \\ B \arrow[r, "\eta_B"] & HI(B) \end{tikzcd}$$

By Proposition 1.5, $f \in \mathcal{M}_I$ if and only if (3.6) is a pullback, that is, α is an isomorphism in the following pullback diagram.

(3.7)
$$\begin{tikzcd} A \arrow[dr, "\alpha"] \arrow[drr, "\eta_A"] \arrow[ddr, "f"'] & & \\ & B \times_{HI(B)} HI(A) \arrow[r] \arrow[d] & HI(A) \arrow[d, "HI(f)"] \\ & B \arrow[r, "\eta_B"] & HI(B) \end{tikzcd}$$

The homomorphism α is surjective if and only if, for each $b \in B$ and for each $a \in A$, such that $b \sim_B f(a)$, there exists $c \in A$ for which

$f(c) = b$ and $c \sim_A a$.

The homomorphism α is injective if and only if for all $a, a^* \in A$, if $f(a) = f(a^*)$ and $a \sim_A a^*$ then, $a = a^*$.

□

EXAMPLE 3.6. Consider the reflection $H \vdash I : \mathbf{Band} \to \mathbf{SLat}$ and let S be a band as in the following multiplication table.

(3.8)

$\bullet$	a	b	q	r
a	a	a	a	a
b	b	b	b	b
r	a	a	r	r
q	b	b	q	q

First notice that $a = aba$, $b = bab$, $q = qrq$, $r = rqr$, $q \neq qaq$. Hence, S is a semilattice of two rectangular bands.

The homomorphism $m : S \to S$, given by $m(a) = m(r) = a$, $m(b) = m(q) = b$ belongs to $\mathcal{M}_I$, because $m_{|_{[a]_{\sim_S}}} : [a]_{\sim_S} \to [a]_{\sim_S}$ and $m_{|_{[r]_{\sim_S}}} : [r]_{\sim_S} \to [a]_{\sim_S}$ are isomorphisms.

Consider the reflection $H \vdash I : \mathbf{CommSgr} \to \mathbf{SLat}$. Let $F(a)$ be the free commutative semigroup on $\{a\}$ and let $F(a,\ b)$ be the free commutative semigroup on $\{a,\ b\}$. $F(a)$ is archimedean, while $F(a,\ b)$ has three archimedean classes, which are $F(a)$, $F(b)$, and $K = \{a^n b^m \mid n,\ m \in \mathbf{N}\}$.

The inclusion homomorphism $i : F(a) \to F(a,\ b)$, $i(a) = a$, belongs to $\mathcal{M}_I$.

In general, given a reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ of a variety of universal algebras $\mathbb{C}$ into an idempotent subvariety $\mathbb{M}$, the inclusion homomorphism of one of the congruence classes of any $S \in \mathbb{C}$ into S belongs to $\mathcal{M}_I$.

For a simple reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ from a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$ the $(\mathcal{E}_I, \mathcal{M}_I)$-factorization $h = me$ of any homomorphism $h : C \to D$, in $\mathbb{C}$ is given in the following pullback diagram, by Definition 1.4.

$$\begin{tikzcd}
C \arrow[dr, "e"] \arrow[drr, "\eta_C"] \arrow[ddr, "h"'] & & \\
& D \times_{HI(D)} HI(C) \arrow[r] \arrow[d, "m"] & HI(C) \arrow[d, "HI(h)"] \\
& D \arrow[r, "\eta_D"] & HI(D)
\end{tikzcd} \tag{3.9}$$

where:

- $D \times_{HI(D)} HI(C) = \{(d, [c]_{\sim_C}) \in D \times HI(C) \mid [d]_{\sim_D} = [h(c)]_{\sim_D}\}$;
- $e : C \to D \times_{HI(D)} HI(C)$;
  $c \mapsto (h(c), [c]_{\sim_C})$
- $m : D \times_{HI(D)} HI(C) \to D$.
  $(d, [c]_{\sim_C}) \mapsto d$

3.2. Stably-vertical morphisms in universal algebras

In this section we will state necessary conditions for a vertical homomorphism to be stably-vertical. Then, we characterize the class $\mathcal{E}'_I$ of stably-vertical morphisms, for some reflections $H \vdash I : \mathbb{C} \to \mathbb{M}$ from a variety of universal algebras $\mathbb{C}$ into an idempotent subvariety $\mathbb{M}$.

- Let $H \vdash I : \mathbb{C} \to \mathbb{M}$ be a reflection from a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$;
- let $\mathcal{E}'_I$ denote the largest subclass of $\mathcal{E}_I$ which is closed under pullbacks, called the class of stably-vertical morphisms;
- let $\langle x \rangle_C$ denote the subalgebra of C, generated by $x \in C$, $C \in \mathbb{C}$;
- let $\mathcal{F} = \{e : X \to Y \in \mathbb{C} \mid \forall_{y \in Y}\ \langle y \rangle_Y \cap e(X) \neq \emptyset\}$ where $e(X)$ is the homomorphic image of X, by e [1];

[1]this notation $\mathcal{F} = \{e : X \to Y \in \mathbb{C} \mid \forall_{y \in Y}\ \langle y \rangle_Y \cap e(X) \neq \emptyset\}$ will be used forward in this text

- consider the factorization system

$(\mathcal{E},\mathcal{M}) =$ (Surjective homomorphisms, Injective homomorphisms)[2], on $\mathbb{C}$:

$$(3.10)\qquad \begin{tikzcd} C \arrow[rr, "h"] \arrow[dr, "h_{|C}"'] & & D \\ & h(C) \arrow[ur, "\subseteq"'] & \end{tikzcd}$$

Definition 3.7. A homomorphism $e : A \to B$ belongs to $\mathcal{E}'_I$ if and only if its pullback along any homomorphism $f : C \to B$ belongs to $\mathcal{E}_I$.

The next Proposition 3.8 states a necessary condition for a homomorphism to be stably-vertical.

Proposition 3.8. *For any reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ from a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$, if $e : C \to D$ lies in $\mathcal{E}'_I$ then, $\langle d\rangle_D \cap e(C) \neq \emptyset$, for every $d \in D$.*

Proof. Consider the following pullback diagram

$$(3.11)\qquad \begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & C \arrow[d, "e"] \\ \langle d\rangle_D \arrow[r, "\subseteq"] & D \end{tikzcd}$$

where $e \in \mathcal{E}'_I$.

Suppose that $\langle d\rangle_D \cap e(C) = \emptyset$, for some $d \in D$ then, $P = \emptyset$, and $\eta_P(P) = I(P) = \emptyset$. Since $\langle d\rangle_D \neq \emptyset$ then, $\eta_{\langle d\rangle_D}(\langle d\rangle_D) = I(\langle d\rangle_D) \neq \emptyset$, and $I(\pi_1) : I(P) \to I(\langle d\rangle_D)$ is not an isomorphism, which contradicts the assumption of e belonging to $\mathcal{E}'_I$. $\square$

[2]this notation

$$\mathcal{E} = \{\text{Surjective homomorphisms}\} \quad \mathcal{M} = \{\text{Injective homomorphisms}\}$$

will be used forward in this text

REMARK 3.9. By Proposition 3.8 we conclude that $\mathcal{E}'_I \subseteq \mathcal{F}$. On the other hand, $\mathcal{E}'_I \subseteq \mathcal{E}_I$. Therefore, $\mathcal{E}'_I \subseteq \mathcal{E}_I \cap \mathcal{F}$.

PROPOSITION 3.10. *If every element of any $M \in \mathbb{M}$ is a subalgebra then, the following two conditions are equivalent:*

(a) *$I(\pi_1)$ and $I(\pi_2)$ are jointly monic for all the pullback diagrams in $\mathbb{C}$*

$$\begin{tikzcd} A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ A \arrow[r] & C \end{tikzcd} , \tag{3.12}$$

such that $g \in \mathcal{E}_I \cap \mathcal{F}$;

(b) $\mathcal{E}'_I = \mathcal{E}_I \cap \mathcal{F}$.

PROOF. (a)$\Rightarrow$(b):

Consider the pullback diagram

$$\begin{tikzcd} I(A \times_C B) \arrow[dr, "w"] \arrow[drr, "I(\pi_2)"] \arrow[ddr, "I(\pi_1)"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & I(B) \arrow[d, "I(g)"] \\ & I(A) \arrow[r, "I(f)"] & I(C) \end{tikzcd} \tag{3.13}$$

where f is an arbitrary homomorphism, $g \in \mathcal{E}_I \cap \mathcal{F}$, $P = I(A) \times_{I(C)} I(B)$ and w is the unique morphism making the diagram commute.

Since $I(\pi_1)$ and $I(\pi_2)$ are jointly monic, w is a monomorphism, which in a variety of universal algebras is an injective homomorphism.

Since $g \in \mathcal{E}_I$, $I(g)$ is an isomorphism. Hence, p_1 is also an isomorphism. Therefore, $I(\pi_1) = p_1 \circ w$ is an injective homomorphism.

On the other hand, let $[l]_{\sim_A}$ be an arbitrary class of $A/{\sim_A} = HI(A)$. Since every element of any M in $\mathbb{M}$ is a subalgebra, and $\eta_{\langle l\rangle_A} : \langle l\rangle_A \to HI(\langle l\rangle_A)$ is a homomorphism of universal algebras, $HI(\langle l\rangle_A) = \eta_{\langle l\rangle_A}(\langle l\rangle_A) = \langle \eta_{\langle l\rangle_A}(l)\rangle_{HI(A)} = \mathbf{1}$. Therefore, $\langle l\rangle_A \subseteq [l]_{\sim_A}$.

Since $f : A \to C$ is a homomorphism of universal algebras, $f(\langle l\rangle_A) = \langle f(l)\rangle_C$. Since $g \in \mathcal{F}$, $\langle f(l)\rangle_C \cap g(B) \neq \emptyset$, which implies $\langle l\rangle_A \cap \pi_1(A \times_C B) \neq \emptyset$. Hence, $[l] \sim_A \cap \pi_1(A \times_C B) \neq \emptyset$. Therefore, $I(\pi_1)$ is also surjective and so, $g \in \mathcal{E}'_I$. Thus, $\mathcal{E}'_I \supseteq \mathcal{E}_I \cap \mathcal{F}$. By Remark 3.9, $\mathcal{E}'_I \subseteq \mathcal{E}_I \cap \mathcal{F}$.

(b)⇒(a):

Consider the pullback diagram (3.13), where $g \in \mathcal{E}_I \cap \mathcal{F}$. Since $\mathcal{E}'_I = \mathcal{E}_I \cap \mathcal{F}$, by hypothesis, $I(\pi_1)$ and $I(g)$ are isomorphisms. Hence, the outside square is a pullback, and $I(\pi_1)$, $I(\pi_2)$ are jointly monic.

□

Corollary 3.11. *Suppose that every element of any $M \in \mathbb{M}$ is a subalgebra and that $I(\pi_1)$ and $I(\pi_2)$ are jointly monic in the following pullback diagram in $\mathbb{C}$.*

(3.14)
$$\begin{tikzcd} A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "g"] \\ A \arrow[r] & C \end{tikzcd}$$

Then, $g \in \mathcal{E}'_I$ if and only if $g \in \mathcal{E}_I \cap \mathcal{F}$.

Example 3.12. Consider the reflection $H \vdash I$: **CommSgr** $\to$ **SLat** and the following pullback diagram in **CommSgr**,

$$\begin{tikzcd} A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "e"] \\ A \arrow[r, "f"] & C \end{tikzcd} \tag{3.15}$$

It is well known that a commutative semigroup is semilattice indecomposable if and only if it is archimedean and a commutative semigroup is a semilattice of archimedean commutative subsemigroups, which are its components in the reflection **CommSgr** $\to$ **SLat**.

In other words: Let C be a commutative semigroup and let $a, b \in C$. Then, $a \sim_C b$ if and only if there exist $m, n \in \mathbf{N}$ and there exist $c, d \in C$, such that $a^m = bc$ and $b^n = ad$.

(A) $I(\pi_1)$ and $I(\pi_2)$ are jointly monic, provided C has cancellation law.

Let $(a_1, b_1),\ (a_2, b_2) \in A \times_C B$ be such that $a_1 \sim_A a_2$ and $b_1 \sim_B b_2$, that is, there exist $m, n, p, q \in \mathbf{N}$; $c,\ d \in A$; $u,\ v \in B$, such that $a_1^m = a_2 c$; $a_2^n = a_1 d$; $b_1^p = b_2 u$; $b_2^q = b_1 v$.

We need:
$(x, y),\ (z, t) \in A \times_C B$ and $r,\ s \in \mathbf{N}$ such that
$(a_1, b_1)^r = (a_2, b_2)(x, y)$ and $(a_2, b_2)^s = (a_1, b_1)(z, t)$.

We take $(x, y) = (a_2^{p-1} c^p,\ b_2^{m-1} u^m),\ (z, t) = (a_1^{q-1} d^q,\ b_1^{n-1} v^n)$ and $r = mp,\ s = nq$.

Clearly we have $(a_1, b_1)^r = (a_1^r, b_1^r) = (a_2, b_2)(x, y)$ and $(a_2, b_2)^s = (a_2^s, b_2^s) = (a_1, b_1)(z, t)$.

We have to prove:

(1) $f(x) = e(y)$,

(2) $f(z) = e(t)$.

(1) would follow from

$f(a_2)f(x) = f(a_2^p c^p) = f(a_1^{mp}) = e(b_1^{mp}) = e(b_2^m u^m) = e(b_2)e(y)$, by cancellation.

(2) would follow from

$f(a_1)f(z) = f(a_1^q d^q) = f(a_2^{nq}) = e(b_2^{nq}) = e(b_1^n v^n) = e(b_1)e(t)$, by cancellation.

(B) $I(\pi_1)$ and $I(\pi_2)$ are jointly monic, provided C is a finitely generated commutative semigroup.

(I) It is known (cf. Proposition 9.6 in [**4**] p.136) that if C is a finitely generated commutative semigroup, any of its archimedean components, A is such that $A^k = \{a \in C \mid a = a_1 ... a_k,$ with $a_1, , a_k \in A\}$ is cancellative, for some $k \in \mathbf{N}$.

(II) Since the subvariety of semilattices is idempotent, each archimedean component of the semilattice decomposition of a commutative semigroup S is a subalgebra of S. Therefore, every power of each archimedean component is a subsemigroup of S.

(III) It is known (see [**12**]) that any archimedean component of the semilattice decomposition of a commutative semigroup is semilattice indecomposable.
Let S be a commutative semigroup and consider an archimedean class $[x]_{\sim_S}$ in S. Then, for every $z, y \in [x]_{\sim_S}$, there exist $m, n \in \mathbf{N}$ and there exist $c, d \in [x]_{\sim_S}$, such that $z^m = yc$ and $y^n = zd$, since $I([x]_{\sim S}) = \mathbf{1}$.

Let C be a finitely generated commutative semigroup.

Let $(a_1, b_1),\ (a_2, b_2) \in A \times_C B$ be such that $a_1 \sim_A a_2$ and $b_1 \sim_B b_2$.

Let $k \in \mathbf{N}$ be such that $[f(a_1)]^k_{\sim_C} = [e(b_1)]^k_{\sim_C}$ is cancellative, by (I).

Consider the following pullback (3.16), where

- $H = [f(a_1)]^k_{\sim_C} = [e(b_1)]^k_{\sim_C}$,
- $X = [a_1]^k_{\sim_A} = [a_2]^k_{\sim_A}$,
- $Y = [b_1]^k_{\sim_B} = [b_2]^k_{\sim_B}$,
- $f_{|X} : X \to H$ and $e_{|Y} : Y \to H$ are restrictions, respectively, of $f : A \to C$ and $e : B \to C$.

Notice that:

(a) $a_1^k,\ a_2^k \ \in\ X$ and $b_1^k,\ b_2^k \ \in\ Y$;

(b) $f(X) \subseteq H$ and $e(Y) \subseteq H$, since $f([a_1]_{\sim_A}) \subseteq [f(a_1)]_{\sim_C}$ and $e([b_1]_{\sim_B}) \subseteq [e(b_1)]_{\sim_C}$.

(3.16)
$$\begin{tikzcd} X \times_H Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "e_{|Y}"] \\ X \arrow[r, "f_{|X}"] & H \end{tikzcd}$$

We need:

(1) $(a_1^k, b_1^k),\ (a_2^k, b_2^k) \ \in\ X \times_H Y$;

(2) $a_1^k \sim_X a_2^k$ and $b_1^k \sim_Y b_2^k$.

(1) would follow from:

$f(a_1) = e(b_1)$ and $f(a_2) = e(b_2)$, since $(a_1, b_1),\ (a_2, b_2) \ \in A \times_C B$. Hence, $f(a_1^k) = e(b_1^k)$ and $f(a_2^k) = e(b_2^k)$. Therefore $f_{|X}(a_1^k) = e_{|Y}(b_1^k)$ and $f_{|X}(a_2^k) = e_{|Y}(b_2^k)$, that is, $(a_1^k, b_1^k),\ (a_2^k, b_2^k) \ \in X \times_H Y$.

(2) would follow from:

There exist $m, n, p, q \in \mathbf{N}$ and there exist $c, d \in [a_1]_{\sim_A}$, $u, v \in [b_1]_{\sim_B}$, such that $a_1^m = a_2c$, $a_2^n = a_1d$, $b_1^p = b_2u$, $b_2^q = b_1v$, by (III).

Therefore, $(a_1^k)^m = a_2^kc^k$, $(a_2^k)^n = a_1^kd^k$, $(b_1^k)^p = b_2^ku^k$, $(b_2^k)^q = b_1^kv^k$, with $c^k, d^k \in X$ and $u^k, v^k \in Y$.

Hence, $(a_1^k, b_1^k) \sim_{X\times_H Y} (a_2^k, b_2^k)$, by (A), since H is cancellative.

Thus, $(a_1^k, b_1^k) \sim_{A\times_C B} (a_2^k, b_2^k)$, since $X \times_H Y$ is a subsemigroup of $A \times_C B$.

Therefore, $(a_1, b_1) \sim_{A\times_C B} (a_2, b_2)$, since $(a_1^k, b_1^k) \sim_{A\times_C B} (a_1, b_1)$ and $(a_2^k, b_2^k) \sim_{A\times_C B} (a_2, b_2)$.

(C) If C is any commutative semigroup then $I(\pi_1)$ and $I(\pi_2)$ are jointly monic.

Let (a_1, b_1), $(a_2, b_2) \in A \times_C B$ be such that $a_1 \sim_A a_2$ and $b_1 \sim_B b_2$.

That is, there exist $m, n, p, q \in \mathbf{N}$; c, $d \in A$; u, $v \in B$, such that $a_1^m = a_2c$; $a_2^n = a_1d$; $b_1^p = b_2u$; $b_2^q = b_1v$.

Consider the finitely generated subsemigroup H of C:

$H = \langle f(a_1), f(a_2), f(c), f(d), e(u), e(v)\rangle_C$.

Recall that $f(a_1) = e(b_1)$, $f(a_2) = e(b_2)$ and consider the following pullback, where $X = f^{-1}(H)$, $Y = e^{-1}(H)$ and $f_{|X} : X \to H$, $e_{|Y} : Y \to H$ are restrictions, respectively, of $f : A \to C$ and $e : B \to C$.

$$\begin{tikzcd} X \times_H Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "e_{|Y}"] \\ X \arrow[r, "f_{|X}"] & H \end{tikzcd} \tag{3.17}$$

Notice that:

(j) $a_1, a_2, c, d \in X$ and $b_1, b_2, u, v \in Y$;

(jj) $(a_1, b_1), (a_2, b_2) \in X \times_H Y$;

(jjj) $a_1 \sim_X a_2$ and $b_1 \sim_Y b_2$, since $c, d \in X$ and $u, v \in Y$.

Hence, by (B), $(a_1, b_1) \sim_{X \times_H Y} (a_2, b_2)$.

Therefore, $(a_1, b_1) \sim_{A \times_C B} (a_2, b_2)$, since $X \times_H Y$ is a subsemigroup of $A \times_C B$.

In conclusion, in the reflection **CommSgr** $\to$ **SLat**, $I(\pi_1)$ and $I(\pi_2)$ are jointly monic for every pullback diagram.
Then, $\mathcal{E}'_I = \mathcal{E}_I \cap \mathcal{F}$.

An example of a homomorphism that belongs to $\mathcal{E}'_I$ is the inclusion homomorphism of the semigroup of positive integers, $\mathbf{N}$ into the semigroup of positive rational numbers, $\mathbf{Q}^+$.
Another example of a homomorphism that belongs to $\mathcal{E}'_I$, if C is a finite group, with identity e, is the inclusion homomorphism of the group $\{e\}$ into C.
Another example of a homomorphism that belongs to $\mathcal{E}'_I$, if C has a subgroup H such that every $c \in C$ has a power in H, is the inclusion homomorphism of H into C.

In the following Remark 3.13 we exhibit, for the reflection $H \vdash I$: **CommSgr** $\to$ **SLat**, a homomorphism that belongs to $\mathcal{E}_I$, but does not belong to $\mathcal{E}'_I$. Thus, $\mathcal{E}'_I \neq \mathcal{E}_I$, for this reflection.

Remark 3.13. Let $\mathbf{Q}^+$, $\mathbf{R}^+$, be the additive commutative semigroups of positive rational, and positive real numbers, respectively, and consider the pullback diagram

$$
\begin{tikzcd}
P \arrow[r] \arrow[d] & \mathbf{Q}^+ \arrow[d, "e"] \\
FU(\mathbf{R}^+) \arrow[r, "\varepsilon_{\mathbf{R}^+}"] & \mathbf{R}^+
\end{tikzcd} \quad , \tag{3.18}
$$

where $e : \mathbf{Q}^+ \to \mathbf{R}^+$ is the inclusion homomorphism and $\varepsilon_{\mathbf{R}^+}$ is the counit morphism in the free adjunction $U \vdash F : \mathbf{Set} \to \mathbf{CommSgr}$. The morphism e lies in $\mathcal{E}_I$, as a consequence of $\mathbf{Q}^+$ and $\mathbf{R}^+$ being archimedean, but $e \notin \mathcal{E}'_I$ because $[\sqrt{2}]_{\sim_{FU(\mathbf{R}^+)}} \cap \pi_1(P) = \emptyset$.

Notice however that $\mathcal{E}'_I = \mathcal{E}_I$, for a reflection $H \vdash I : \mathbb{S} \to \mathbf{SLat}$, where $\mathbb{S}$ is a subvariety of $\mathbf{CommSGr}$ which satisfies $x^k = x^{k+p}$, with $k, p \in \mathbf{N}$, as in the next Remark 3.14.

Remark 3.14. There are some subvarieties $\mathbb{S}$ of $\mathbf{CommSGr}$, namely those that satisfy $x^k = x^{k+p}$, with $k, p \in \mathbf{N}$, such that, $\mathcal{E}'_I = \mathcal{E}_I$, for the reflection $H \vdash I : \mathbb{S} \to \mathbf{SLat}$:

Let $a, b \in S$, for $S \in \mathbb{S}$. If $a \sim_S b$ then, there exist $i, j \in \mathbf{N}$ and $c, d \in S$, such that $a^i = bc$ and $b^j = ad$, which implies:

$$a^{ij^2} = a^j d^j c^{j^2} \quad \text{and} \quad b^{ij^2} = ad^{ij+1}c^j \tag{3.19}$$

There exist $l, r \in \mathbf{N}$, with $r < p$, such that $lp = k + r$. for $k, p \in \mathbf{N}$. From 3.19, we can write

$$a^{ij^2 lp} = a^{jlp} d^{jlp} c^{j^2 lp};$$
$$b^{ij^2 lp} = a^{lp} d^{(ij+1)lp} c^{jlp}.$$

On the other hand, since $x^{k+np+r} = x^{k+qp+r}$, for $q, n \in \mathbf{N_0}$, $r \in \mathbf{N}$, $r < p$:

$$a^{jlp} = a^{(j-1)lp+k+r} = a^{k+r} = a^{lp};$$
$$d^{jlp} = d^{(j-1)lp+k+r} = d^{ijlp+k+r} = d^{(ij+1)lp};$$
$$c^{j^2 lp} = c^{(j^2-1)lp+k+r} = c^{(j-1)lp+k+r} = c^{jlp}.$$

Hence, if $a \sim_S b$ then there exists $m \in \mathbf{N}$ such that $a^m = b^m$.

Conversely, if there exists $m \in \mathbf{N}$ such that $a^m = b^m$, then $a \sim_S b$, since $a \sim_S a^m$ and $b \sim_S b^m$.

$\mathcal{E}'_I = \mathcal{E}_I$:

Consider the pullback diagram

$$\begin{tikzcd} H \times_L S \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & S \arrow[d, "e"] \\ H \arrow[r, "f"] & L \end{tikzcd} \quad , \tag{3.20}$$

where $e \in \mathcal{E}_I$.

$I(\pi_1)$ is injective, because $I(\pi_1)$ and $I(\pi_2)$ are jointly monic(cf. (C) Example 3.12).

$I(\pi_1)$ is surjective as we are going to prove. Let $[h] \sim_H$ be an arbitrary class of $H/ \sim_H$. Since $e \in \mathcal{E}_I$, $[f(h)] \sim_H \cap\, e(S) \neq \emptyset$. Let $s \in S$, be such that $e(s) \sim_L f(h)$. Then, there exists $q \in \mathbf{N}$, such that $(e(s))^q = (f(h))^q$. Therefore, $(h^q, s^q) \in H \times_L S$. Since $h^q \sim_H h$, $[h] \sim_H \cap\, \pi_1(H \times_L S) \neq \emptyset$.

Considering, once again, the reflection $H \vdash I : \mathbf{CommSgr} \to \mathbf{SLat}$, the next Remark 3.15 exhibits a homomorphism that belongs to $\mathcal{E}'_I$, but is not a surjective homomorphism. Thus, $\mathcal{E}'_I = \mathcal{E}_I \cap \mathcal{F} \neq \mathcal{E}_I \cap \mathcal{E}$(cf. Example 3.12).

REMARK 3.15. $\mathcal{E}'_I$ is not contained in $\mathcal{E}$. Hence, $\mathcal{E}'_I \neq \mathcal{E}_I \cap \mathcal{E}$:

Let $d : \mathbf{N} \to \mathbf{N}$ be the semigroup homomorphism that maps n to $2n$, for every $n \in \mathbf{N}$.

$d \in \mathcal{E}'_I$, since $d \in \mathcal{E}_I \cap \mathcal{F}$, but it is not a surjective homomorphism, i.e., $d \notin \mathcal{E}$.

COROLLARY 3.16. *If every element of any $C \in \mathbb{C}$ is a subalgebra then, $\mathcal{E}'_I \subseteq \mathcal{E}_I \cap \mathcal{E}$, and the following conditions are equivalent:*

(a) *$I(\pi_1)$ and $I(\pi_2)$ are jointly monic for all pullback diagrams in $\mathbb{C}$*

$$\begin{tikzcd} A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "g"] \\ A \arrow[r] & C \end{tikzcd} , \tag{3.21}$$

such that $g \in \mathcal{E}_I \cap \mathcal{E}$;

(b) $\mathcal{E}'_I = \mathcal{E}_I \cap \mathcal{E}$.

PROOF. Since $\langle x \rangle_C = x$, for every element x of any $C \in \mathbb{C}$, $\mathcal{F} = \mathcal{E}$. Hence, by Proposition 3.8, $\mathcal{E}'_I \subseteq \mathcal{E}_I \cap \mathcal{E}$.

(a)$\Rightarrow$(b):

If every element of any $C \in \mathbb{C}$ is a subalgebra then, so it is every element of any $M \in \mathbb{M}$. On the other hand, $\mathcal{F} = \mathcal{E}$. Hence, by Proposition 3.10, $\mathcal{E}'_I = \mathcal{E}_I \cap \mathcal{E}$.

(b)$\Rightarrow$(a):

It follows immediately, by considering the pullback diagram (3.21), where $g \in \mathcal{E}_I \cap \mathcal{E}$.

$\square$

EXAMPLE 3.17. Consider the reflection of **Band** into **SLat**. Recall that a band is semilattice indecomposable if and only if it is a rectangular band.

A band is a semilattice of rectangular bands, which are its components in the reflection **Band** into **SLat**.

In other words: Let B be a band and $b,\ c \in B$ then $b \sim_B c$ if and only if $b = bcb$ and $c = cbc$.

Therefore, for the following pullback (3.22), where $f,\ g$ are arbitrary homomorphisms, if $(a, c),\ (x, y) \in A \times_C B$ and $a \sim_A x,\ c \sim_B y$ then, $(a, c) \sim_{A \times_C B} (x, y)$. Because if $a = axa,\ x = xax,\ c = cyc$ and $y = ycy$, then $(a, c) = (a, c)(x, y)(a, c)$ and $(x, y) = (x, y)(a, c)(x, y)$, that is, $I(\pi_1)$ and $I(\pi_2)$ are jointly monic.

$$\begin{tikzcd} A \times_C B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd} \tag{3.22}$$

Hence, by Corollary 3.16 $\mathcal{E}'_I = \mathcal{E}_I \cap \mathcal{E}$.

An example of a homomorphism that belongs to $\mathcal{E}'_I$ is the (unique) homomorphism $e' : S \to \{a\}$, where S is the band in Example 3.4. On the other hand the vertical homomorphism in Example 3.4 is not stably vertical.

Remark 3.18. Under the equivalent conditions, (a) and (b), of Proposition 3.10 the reflection $\langle I, H, \eta, \nu \rangle : \mathbb{C} \to \mathbb{M}$ has stable units: Since the inclusion functor is always faithful and in the case of reflections into subvarieties of universal algebras is always full, all the components of the counit $\nu : IH \to \mathbf{1}_{\mathbb{M}}$ are isomorphisms.

Since $\nu_{I(C)} \circ I\eta_C = 1_{I(C)}$, $I\eta_C$ is an isomorphism, for every C in $\mathbb{C}$. Hence, $\eta_C \in \mathcal{E}_I$, for every $C \in \mathbb{C}$.

On the other hand $\eta_C : C \to C/\sim_C$ is a surjective homomorphism, for every C in $\mathbb{C}$.

Therefore, $\eta_C \in \mathcal{E}_I \cap \mathcal{F}$, for every C in $\mathcal{C}$.

Consider the pullback diagram (1.6). Since $\eta_C \in \mathcal{E}'_I$, $\pi_2 \in \mathcal{E}_I$. Hence $I(\pi_2)$ and $I(\eta_C)$ are both isomorphisms. Therefore, the following commutative square is a pullback

$$\begin{tikzcd} I(C \times_{HI(C)} D) \arrow[r, "I(\pi_2)"] \arrow[d, "I(\pi_1)"] & I(D) \arrow[d, "I(g)"] \\ I(C) \arrow[r, "I(\eta_C)"] & IHI(C) \end{tikzcd} \tag{3.23}$$

Hence, I preserves all pullbacks of the form (1.6).

3.3. Stably-vertical morphisms in a more general setting

In this section we generalize some results of the last section for reflections $H \vdash I : \mathbb{C} \to \mathbb{M}$ from a finitely complete category $\mathbb{C}$ into a full subcategory $\mathbb{M}$, when there exists a functor $U : \mathbb{C} \to \mathbf{Set}$, which preserves finite limits and reflects isomorphisms.

Under certain data, Corollary 3.16 can be generalized for some reflections $H \vdash I : \mathbb{A} \to \mathbb{B}$, of a finitely complete category $\mathbb{A}$ into a full subcategory $\mathbb{B}$.

Assume the following data (3.24):

(1) Let $\mathbb{A}$ be a finitely complete category, and let $\mathbb{B}$ be a full reflective subcategory of $\mathbb{A}$, the inclusion functor being $H : \mathbb{B} \to \mathbb{A}$, with unit $\eta : \mathbf{1}_{\mathbb{A}} \to HI$.

(2) Define the prefactorization system $(\mathcal{E}_I, \mathcal{M}_I)$, by setting $\mathcal{E}_I = (H(\mathrm{mor}\mathbb{B}))^{\uparrow}$, $\mathcal{M}_I = (H(\mathrm{mor}\mathbb{B}))^{\uparrow\downarrow}$. Then, a morphism $f \in \mathcal{E}_I$ if and only if $I(f)$ is an isomorphism(cf. section 1.2).

(3) Let $U : \mathbb{A} \to \mathbf{Set}$ be such that:

  (a) $U(\eta_A)$ is a surjection, for all $A \in \mathbb{A}$;

  (b) U preserves finite limits;

  (c) U reflects isomorphism.

(4) Let $\mathcal{G} = \{f : D \to E \mid U(f) \text{ is a surjection}\}$ (this notation will be used forward in this text).

Propositions 3.19 and 3.20 generalize Corollary 3.16.

Proposition 3.19. *Under* (1), (2), (3) (a), (b) *and* (4) *of data* (3.24), *if* $U_{T,A} : \mathbb{A}(T, A) \to \mathbf{Set}(\{*\}, U(A))$ *is a surjection for all* $A \in \mathbb{A}$, *with* T *a terminal object in* $\mathbb{A}$, *then* $\mathcal{E}'_I \subseteq \mathcal{E}_I \cap \mathcal{G}$.

Proof. Since $\mathcal{E}'_I \subseteq \mathcal{E}_I$ it remains to show that $\mathcal{E}'_I \subseteq \mathcal{G}$. Let $e : B \to C$ lie in $\mathcal{E}'_I$. For an arbitrary $x \in U(C)$ consider the inclusion map $f_x : \{x\} \to U(C)$. By hypothesis, there exists a morphism $m : T \to C$, such that $U(m) = f_x$. Hence, in the pullback diagram

(3.25)
$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "e"] \\ T \arrow[r, "m"] & C \end{tikzcd},$$

$\pi_1 \in \mathcal{E}_I$. Therefore, $I(\pi_1)$ is an isomorphism.

Consider the commutative square

(3.26)
$$\begin{tikzcd} U(P) \arrow[r, "U(\eta_P)"] \arrow[d, "U(\pi_1)"'] & UHI(P) \arrow[d, "UHI(\pi_1)"] \\ U(T) \arrow[r, "U(\eta_T)"] & UHI(T) \end{tikzcd}$$

Since $U(T) = \{x\} \neq \emptyset$, $UHI(T) \neq \emptyset$. Since $HI(\pi_1)$ is an isomorphism, $UHI(\pi_1)$ is a bijective map. Hence, $UHI(P) \neq \emptyset$. Since $U(\eta_P)$ is a surjection, $U(P) \neq \emptyset$. Therefore, $U(e)$ is a surjective map, as a consequence of x being any element of $U(C)$.

$\square$

PROPOSITION 3.20. *Under data* (3.24) (1), (2), (3) (a), (b), (c) *and* (4), *if* $U_{T,A} : \mathbb{A}(T, A) \to \mathbf{Set}(\{*\}, U(A))$ *is a surjection for all* $A \in \mathbb{A}$, *with* T *a terminal object in* $\mathbb{A}$ *then, the following conditions are equivalent:*

(a) $I(\pi_1)$ *and* $I(\pi_2)$ *are jointly monic for all pullbacks in* $\mathbb{A}$

(3.27)
$$\begin{tikzcd} A \times_C B \arrow[r, "p_2"] \arrow[d, "p_1"] & B \arrow[d, "e"] \\ A \arrow[r] & C \end{tikzcd},$$

where $e \in \mathcal{E}_I \cap \mathcal{G}$;

(b) $\mathcal{E}'_I = \mathcal{E}_I \cap \mathcal{G}$.

PROOF.
(a)$\Rightarrow$ (b):

Consider the pullback diagrams (3.28) and (3.29):

(3.28)
$$\begin{tikzcd} U(A\times_C B) \arrow[r, "U(\pi_2)"] \arrow[d, "U(\pi_1)"'] & U(B) \arrow[d, "U(e)"] \\ U(A) \arrow[r] & U(C) \end{tikzcd} \quad ,$$

(3.29)
$$\begin{tikzcd} HI(A\times_C B) \arrow[dr, "w"] \arrow[drr, "HI(\pi_2)"] \arrow[ddr, "HI(\pi_1)"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & HI(B) \arrow[d, "HI(e)"] \\ & HI(A) \arrow[r] & HI(C) \end{tikzcd} \quad ,$$

where $e \in \mathcal{E}_I \cap \mathcal{G}$. By hypothesis, $I(\pi_1)$ and $I(\pi_2)$ are jointly monic and so w is a monomorphism. Since U preserves finite limits, it preserves monomorphisms. Hence, $U(w)$ is an injective map. Therefore, $UHI(\pi_1)$ is injective as a consequence of $UHI(e)$ and so p_1 being bijections.

On the other hand, since $U(e)$ is surjective, $U(\pi_1)$ is also a surjective map.

Consider the following commutative square

$$
\begin{tikzcd}
U(A \times_C B) \arrow[r, "U(\eta_{A\times_C B})"] \arrow[d, "U(\pi_1)"'] & UHI(A \times_C B) \arrow[d, "UHI(\pi_1)"] \\
U(A) \arrow[r, "U(\eta_A)"] & UHI(A)
\end{tikzcd} \quad . \tag{3.30}
$$

Since $U(\pi_1)$ and $U(\eta_A)$ are surjective maps, so it is $U(\eta_A) \circ U(\pi_1)$. Therefore, $UHI(\pi_1)$ is surjective. Hence, $UHI(\pi_1)$ is a bijective map. Since U reflects isomorphisms, $I(\pi_1)$ is an isomorphism. Hence, $\pi_1 \in \mathcal{E}_I$. Therefore, $\mathcal{E}_I \cap \mathcal{F} \subseteq \mathcal{E}'_I$. On the other hand, by Proposition 3.19, $\mathcal{E}'_I \subseteq \mathcal{E}_I \cap \mathcal{F}$.

(b)⇒(a):

Suppose $\mathcal{E}'_I = \mathcal{E}_I \cap \mathcal{G}$, and consider the pullback diagram (3.29), where $e \in \mathcal{E}_I \cap \mathcal{G}$. Since $I(e)$, and $I(\pi_1)$ are isomorphisms, the outside commutative square is a pullback. Therefore, $I(\pi_1)$ and $I(\pi_2)$ are jointly monic.

□

Remark 3.21. Under the assumptions and the equivalent conditions (a) and (b) of Proposition 3.20, if $U_{T,A} : \mathbb{A}(T, A) \to \mathbf{Set}(\{*\}, U(A))$ is a surjection for all $A \in \mathbb{A}$, with T a terminal object in $\mathbb{A}$, the reflection $I \dashv H$ has stable units:

It is straightforward to prove by the arguments of Remark 3.18, since $\mathbb{B}$ is a full subcategory of $\mathbb{A}$, and $U(\eta_A)$ is a surjection for every A in $\mathbb{A}$.

Recall that under these conditions, by Proposition 2.13 and Proposition 2.14, we already knew that these reflections have stable units if and only if they are semi-left-exact.

CHAPTER 4

Covering morphisms

In section 4.1 we will describe the class of coverings of B, $\mathcal{M}^*{}_I/B$, with $B \in \mathbb{C}$, for simple=semi-left-exact reflections of a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$, $H \vdash I : \mathbb{C} \to \mathbb{M}$. Under certain conditions, we conclude that the coverings of B are just the trivial coverings of B, that is, $\mathcal{M}^*{}_I/B = \mathcal{M}_I/B$, for a reflection into an idempotent subvariety. For instance, the reflection of bands into semilattices and the reflection of commutative semigroups into semilattices. Then, if $f : A \to B$ is a Galois descent morphism in $\mathbb{C}$, its Galois groupoid is the equivalence relation given by the kernel pair of $I(f)$.

In section 4.2 we will generalize some of these results for reflections $H \vdash I : \mathbb{C} \to \mathbb{M}$ of a finitely complete general category $\mathbb{C}$ into a full subcategory $\mathbb{M}$, when there exists a functor $U : \mathbb{C} \to \mathbf{Set}$ that preserves finite limits, reflects isomorphisms and such that $U(\eta_C)$ is a surjection, for every unit morphism $\eta_C : C \to HI(C)$ in $\mathbb{C}$.

- Let $H \vdash I : \mathbb{C} \to \mathbb{M}$ be a simple reflection of a finitely complete category $\mathbb{C}$ into a full subcategory $\mathbb{M}$.
- Let $\mathcal{M}_I^*$ denote the class of all the covering morphisms.

Under those conditions (see [**2**, §6.1]):

Definition 4.1. A morphism $m \in \mathcal{M}_I^*$ if and only if there exists an effective descent morphism $p : E \to B$ in $\mathbb{C}$, such that $\pi_1 \in \mathcal{M}_I$, in the following pullback diagram,

$$
\begin{tikzcd}
P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "m"] \\
E \arrow[r, "p"] & B
\end{tikzcd}
\tag{4.1}
$$

The following Lemma 4.2 and Proposition 4.3 are known results:

LEMMA 4.2. *Let $\mathbb{C}$ be a category and suppose that there exists an adjunction $\langle F, U, \lambda, \varepsilon \rangle : \mathbb{S} \to \mathbb{C}$, with unit λ, and counit ε.*

If $U(p)$ is a split epimorphism in $\mathbb{S}$, where $p : E \to B$ is a morphism of $\mathbb{C}$, then there exists $f : FU(B) \to E$ in $\mathbb{C}$ such that $\varepsilon_B = p \circ f$.

PROOF. Since $U(p)$ is a split epimorphism in $\mathbb{S}$, there exists a morphism $h : U(B) \to U(E)$, such that

$$U(p) \circ h = 1_{U(B)}. \tag{4.2}$$

Since $\lambda_{U(B)}$ is universal from $U(B)$ to U, there exists a unique $f : FU(B) \to E$, such that the following diagram commutes.

$$
\begin{tikzcd}
U(B) \arrow[r, "\lambda_{U(B)}"] \arrow[dr, "h"'] & UFU(B) \arrow[d, "U(f)"] \\
 & U(E)
\end{tikzcd}
\tag{4.3}
$$

Hence,
$U(p \circ f) \circ \lambda_{U(B)} = U(p) \circ U(f) \circ \lambda_{U(B)}$,
$= U(p) \circ h$, by (4.3)
$= 1_{U(B)}$, by (4.2)
$= U(\varepsilon_B) \circ \lambda_{U(B)}$, because λ, ε are, respectively, the unit and the counit of the adjunction $\langle F, U, \lambda, \varepsilon \rangle$.

Since $\lambda_{U(B)}$ is universal from $U(B)$ to U, $\varepsilon_B = p \circ f$.

$\square$

Proposition 4.3. *Consider a simple reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$, of a category $\mathbb{C}$ into a full subcategory $\mathbb{M}$, such that there exists an adjunction $\langle F, U, \lambda, \varepsilon\rangle : \mathbb{S} \to \mathbb{C}$, with unit λ, and counit ε, such that:*

(1) *ε_B is an effective descent morphism in $\mathbb{C}$, for all $B \in \mathbb{C}$;*

(2) *if p is an effective descent morphism in $\mathbb{C}$ then, $U(p)$ is a split epimorphism in $\mathbb{S}$.*

Then, $m : A \to B \in \mathcal{M}_I^*$ if and only if $\pi_1 \in \mathcal{M}_I$ in the following pullback diagram,

(4.4)
$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "m"] \\ FU(B) \arrow[r, "\varepsilon_B"] & B \end{tikzcd} .$$

$\pi_1 \in \mathcal{M}_I$.

Proof.
($\Leftarrow$)

Since ε_B is an effective descent morphism in $\mathbb{C}$, by (1), and $\pi_1 \in \mathcal{M}_I$ it follows from Definition 4.1.

($\Rightarrow$)

If $m \in \mathcal{M}_I^*$ then, there exists an effective descent morphism (e.d.m.) $p : E \to B$ in $\mathbb{C}$, such that in the pullback diagram (4.1) $\pi_1 \in \mathcal{M}_I$. Since p is an e.d.m., $U(p)$ is a split epimorphism, by (1). Hence, by Lemma 4.2, there exists $f : FU(B) \to E$, such that $\varepsilon_B = p \circ f$.

Consider the diagram

(4.5)
$$\begin{tikzcd} Q \arrow[r, "\beta"] \arrow[d, "f^*(p^*(m))"] & P \arrow[r, "\alpha"] \arrow[d, "p^*(m)"] & A \arrow[d, "m"] \\ FU(B) \arrow[r, "f"'] & E \arrow[r, "p"'] & B \end{tikzcd} ,$$

where both squares are pullbacks. Then, the outside square is also a pullback. On the other hand, since $p^*(m) \in \mathcal{M}_I$, and $\mathcal{M}_I$ is closed under pullbacks, $f^*(p^*(m)) \in \mathcal{M}_I$. That is, the pullback of m along $\varepsilon_B = p \circ f$ belongs to $\mathcal{M}_I$.

□

4.1. Covering morphisms in universal algebras

Consider a reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$, from a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$.

It is known that an effective descent morphism in varieties of universal algebras is just a surjective homomorphism. So, its image by the the underlying functor U is a split epimorphism in **Set**. Then, by Proposition 4.3, a morphism m lies in $\mathcal{M}_I^*$ if and only if $\pi_1 \in \mathcal{M}_I$ in the pullback diagram (4.4), where ε is the counit of the adjunction $\langle F, U, \lambda, \varepsilon\rangle : \mathbf{Set} \to \mathbb{C}$, F being the free functor and U being the underlying functor.

In the reflection $H \vdash I : \mathbf{CommSgr} \to \mathbf{SLat}$ the homomorphisms $\varepsilon_A : FU(A) \to A$, with A any commutative semigroup, satisfy the following property[1]:

If $a \sim_A b$ in A, then there exist $x, y \in FU(A)$, such that $x \sim_{FU(A)} y$ and $\varepsilon_A(x) \in \langle a\rangle_A$, $\varepsilon_A(y) \in \langle b\rangle_A$.

We are going to prove that the property holds for the reflection $H \vdash I : \mathbf{CommSgr} \to \mathbf{SLat}$.

Let A be a commutative semigroup and $a,\ b \in A$.

Recall that $a \sim_A b$ if and only if there exist $n,\ m \in \mathbf{N};\ c,\ d \in A$, such that $a^n = bc$ and $b^m = ad$.

Then, $a^{nm} = b^m c^m = adc^m$ and $b^{mn} = a^n d^n = bcd^n$.

Hence, $a^{n^2 m} = a^n d^n c^{mn} = bcd^n c^{mn} = bc^{mn+1} d^n$.

[1]in fact this property is satisfied by every surjective homomorphisms, that is, by every effective descent morphism in the reflections $H \vdash I : \mathbf{CommSgr} \to \mathbf{SLat}$ and $H \vdash I : \mathbf{Band} \to \mathbf{SLat}$.

Thus, $a^{n^2m} = bc^{nm+1}d^n$ and $b^{nm} = bcd^n$.

The elements $bc^{nm+1}d^n,\ bcd^n \in FU(A)$ are in the same archimedean class of $FU(A)$ and $\varepsilon_A(bc^{nm+1}d^n) \in \langle a \rangle_A,\ \varepsilon_A(bcd^n) \in \langle b \rangle_A$.

This property can be viewed in terms of kernel-pairs of the reflection unit morphisms, as in Remark 4.4.

Remark 4.4. Consider a reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ of a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$ and the following diagram in $\mathbb{C}$, where $w = \langle f \circ p_1,\ f \circ p_2 \rangle$.

(4.6)

$$\begin{tikzcd}
A \times_{HI(A)} A \arrow[rrr, "p_2"] \arrow[dr, "w"] \arrow[ddd, "p_1"'] & & & A \arrow[dl, "f"'] \arrow[ddd, "\eta_A"] \\
& B \times_{HI(B)} B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "\eta_B"] & \\
& B \arrow[r, "\eta_B"] & HI(B) & \\
A \arrow[ur, "f"] \arrow[rrr, "\eta_A"'] & & & HI(A) \arrow[ul, "HI(f)"]
\end{tikzcd}$$

The following conditions are equivalent:

(1) For all $b, b' \in B$, if $b \sim_B b'$ then, there exist $a, a' \in A$ such that $f(a) \in \langle b \rangle_B$, $f(a') \in \langle b' \rangle_B$ and $a \sim_A a'$, where $\langle b \rangle_B,\ \langle b' \rangle_B$ are, respectively, the subalgebra of B generated by $b,\ b' \in B$.

(2) For all $(b, b') \in B \times_{HI(B)} B,\ w(A \times_{HI(A)} A) \cap \langle b \rangle_B \times \langle b' \rangle_B \neq \emptyset$

Notice that, if every element in any $M \in \mathbb{M}$ is a subalgebra and $b \sim_B b'$, then $\langle b \rangle_B \times \langle b' \rangle_B \subseteq B \times_{HI(B)} B$.

Proposition 4.5. *Let $H \vdash I : \mathbb{C} \to \mathbb{M}$ be a simple reflection of a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$, such that:*

- *every element in any $M \in \mathbb{M}$ is idempotent;*
- *for all $s, s' \in S \in \mathbb{C}$, such that $s \sim_S s'$, there exist $w, w' \in FU(S)$, such that $w \sim_{FU(S)} w'$, $\varepsilon_S(w) \in \langle s \rangle_S$ and $\varepsilon_S(w') \in \langle s' \rangle_S$ (where $\varepsilon : FU \to \mathbf{1}_{\mathbb{C}}$ is the counit of the adjunction $\langle F, U, \lambda, \varepsilon \rangle : \mathbf{Set} \to \mathbb{C}$).*

Suppose that in the following pullback diagram $\pi_1 \in \mathcal{M}_I$.

(4.7)
$$\begin{tikzcd}
FU(S) \times_S L \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & L \arrow[d, "f"] \\
FU(S) \arrow[r, "\varepsilon_S"] & S
\end{tikzcd}$$

Then the following conditions are equivalent:

(1) *$I(\pi_1)$ and $I(\pi_2)$ are jointly monic;*

(2) *the reflector I preserves the pullback* (4.7)*.*

Proof. If I preserves the pullback (4.7) then, $I(\pi_1)$ and $I(\pi_2)$ are jointly monic.

On the other hand, suppose that $I(\pi_1)$ and $I(\pi_2)$ are jointly monic. We want to prove that the pullback (4.7) is preserved by I.

Consider the following pullback diagram

(4.8)
$$\begin{tikzcd}
HI(FU(S) \times_S L) \arrow[dr, "\alpha"] \arrow[drr, "HI(\pi_2)"] \arrow[ddr, "HI(\pi_1)"'] & & \\
& HIFU(S) \times_{HI(S)} HI(L) \arrow[r] \arrow[d] & HI(L) \arrow[d, "HI(f)"] \\
& HIFU(S) \arrow[r, "HI(\varepsilon_S)"] & HI(S) .
\end{tikzcd}$$

Since α is an injective homomorphism, it remains to prove that α is also surjective.

It is well known that for varieties of universal algebras the functor $U : \mathbb{C} \to \mathbf{Set}$ preserves finite limits and the unit homomorphisms, η_C, of the reflection are surjective homomorphisms, for every $C \in \mathbb{C}$. Hence, by section 1.5, α is a surjective homomorphism if and only if for every $t \in L$ and $w \in FU(S)$, such that $f(t) \sim_S \varepsilon_S(w)$, there exists $(w', t') \in FU(S) \times_S L$ with $w \sim_{FU(S)} w',\ t \sim_L t'$.

Let $w \in FU(S)$ and $t \in L$, be such that $\varepsilon_S(w) \sim_S f(t)$. There exist $w_1, w_2 \in FU(S)$, such that $\varepsilon_S(w_1) \in \langle \varepsilon_S(w) \rangle_S$, $\varepsilon_S(w_2) \in \langle f(t) \rangle_S$, and $w_1 \sim_{FU(S)} w_2$.

Since $\langle f(t) \rangle_S = f\langle t \rangle_L$, there exists $t' \in \langle t \rangle_L$, such that $\varepsilon_S(w_2) = f(t')$, i.e., $(w_2, t') \in FU(S) \times_S L$. Since $\pi_1 \in \mathcal{M}_I$ and $w_1 \sim_{FU(S)} w_2$, there exists $t^* \in L$, such that $\varepsilon_S(w_1) = f(t^*)$ and $(w_2, t') \sim_{FU(S) \times_S L} (w_1, t^*)$. Hence $t^* \sim_L t'$.

Since every element in any $M \in \mathbb{M}$ is idempotent, $t' \sim_L t$. Therefore, $t^* \sim_L t$.

Since $\varepsilon_S(w_1) \in \langle \varepsilon_S(w) \rangle_S = \varepsilon_S \langle w \rangle_{FU(S)}$, there exists $w^* \in \langle w \rangle_{FU(S)}$, such that $\varepsilon_S(w^*) = \varepsilon_S(w_1)$.

Since $w^* \in \langle w \rangle_{FU(S)}$, and every element in any $M \in \mathbb{M}$ is idempotent, $w^* \sim_{FU(S)} w$, and $\varepsilon_S(w^*) = \varepsilon_S(w_1) = f(t^*)$.

Therefore, there exists $(w^*, t^*) \in FU(S) \times_S L$, such that $w^* \sim_{FU(S)} w$ and $t^* \sim_L t$. Thus, α is also surjective.

$\square$

Corollary 4.6. *Under the equivalent conditions (1) and (2) of Proposition* 4.5, *in the pullback (4.7), $f \in \mathcal{M}_I$ if and only if $\pi_1 \in \mathcal{M}_I$.*

Proof. By Proposition 4.5, the following commutative diagram is a pullback, since both squares inside are pullbacks.

(4.9)
$$\begin{tikzcd}
FU(S)\times_S L \arrow[r, "\eta_{FU(S)\times_S L}"] \arrow[d, "\pi_1"] & HI(FU(S)\times_S L) \arrow[r, "HI(\pi_2)"] \arrow[d, "HI(\pi_1)"] & HI(L) \arrow[d, "HI(f)"] \\
FU(S) \arrow[r, "\eta_{FU(S)}"'] & HIFU(S) \arrow[r, "HI(\varepsilon_S)"'] & HI(S)
\end{tikzcd}$$

Since $\eta_S \circ \varepsilon_S = HI(\varepsilon_S) \circ \eta_{FU(S)}$, the outside square of the following commutative diagram is a pullback.

(4.10)
$$\begin{tikzcd}
FU(S)\times_S L \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & L \arrow[r, "\eta_L"] \arrow[d, "f"] & HI(L) \arrow[d, "HI(f)"] \\
FU(S) \arrow[r, "\varepsilon_S"'] & S \arrow[r, "\eta_S"'] & HI(S)
\end{tikzcd}$$

Since ε_S is an effective descent morphism in $\mathbb{C}$ and the left square is a pullback, the right square is a pullback (cf. Lemma 4.6 in [**2**]).
□

EXAMPLE 4.7. In the reflection $H \vdash I$: **CommSgr** $\to$ **SLat**, $\mathcal{M}^*{}_I = \mathcal{M}_I$, by Proposition 4.5 and Corollary 4.6, since the reflection, $I(\pi_1)$ and $I(\pi_2)$, of the projections, of any pullback are jointly monic, by Example 3.12.

In this reflection, if $f : L \to S$ is a Galois descent homomorphism, then the reflector I preserves the kernel pair of f.

PROPOSITION 4.8. *Under the conditions of Proposition 4.5, if* $f : L \to S$ *in the pullback diagram (4.7) is a Galois descent homomorphism then the following two conditions hold:*

- *the Galois groupoid* $Gal(L, f)$ *of* f *is the equivalence relation given by the kernel pair of* $I(f)$ *in* $\mathbb{M}$;
- $\mathcal{M}_I/S \cong \mathbb{M}^{Gal[f]}$.

PROOF. Suppose that $f : L \to S$ is a Galois descent homomorphism. Then, f is an effective descent homomorphism in $\mathbb{C}$ and $\pi_1 \in \mathcal{M}_I$ in the following pullback diagram.

$$
\begin{tikzcd}
Ker f \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & L \arrow[d, "f"] \\
L \arrow[r, "f"] & S
\end{tikzcd}
\tag{4.11}
$$

Since effective descent morphisms for varieties of universal algebras are just the surjective homomorphisms, $U(f)$ is a split epimorphism in **Set**. Hence, by Lemma 4.2, there exists $m : FU(S) \to L$ in $\mathbb{C}$, such that $\varepsilon_S = f \circ m$. Therefore, the pullback of f along ε_S, $\varepsilon_S^*(f)$ also belongs to $\mathcal{M}_I$. Thus, by Corollary 4.6, $f \in \mathcal{M}_I$. Since the reflection is simple and therefore, semi-left-exact, by Lemma 3.1, I preserves the kernel pair of f.

On the other hand, $\mathcal{M}_I/S = \mathbf{Split}_S(f)$, by Corollary 4.6. The equivalence of categories follows from Theorem 1.31.

□

The following Example 4.9 exhibits a Galois descent homomorphism (which is not an isomorphism), in the reflection $H \vdash I : \mathbf{CommSgr} \to \mathbf{SLat}$.

Example 4.9. Let $S = F(a, b;\ a^2 = a)$ be the semigroup obtained from the free commutative semigroup on two generators, a, b, by the congruence generated by $a^2 = a$. Let $\mathbf{N_0}$ the additive monoid of non-negative integer numbers.

Let $h : S \to \mathbf{N_0}$ be the unique homomorphism such that $h(a) = 0,\ h(b) = 1$. Clearly h is a surjective homomorphism.

The effective descent morphism $h : S \to \mathbf{N_0}$ is a Galois descent homomorphism if and only if it belongs to $\mathcal{M}_I$, since $\mathcal{M}^*{}_I/\mathbf{N_0} = \mathcal{M}_I/\mathbf{N_0}$.

$\mathbf{N_0}$ has two archimedean classes, namely, $\{0\}$ and $\mathbf{N}$, while S has three archimedean classes, namely, $\{a\}$, $F\{b\}$ and $K = \{ab^n \mid n \in \mathbf{N}\}$.

The restrictions $h_{|K} : K \to \mathbf{N}$, $h_{|F\{b\}} : F\{b\} \to \mathbf{N}$, and $h_{|\{a\}} : \{a\} \to \{0\}$ are isomorphisms. Then, $h \in \mathcal{M}_I \cap \mathcal{E}$.

On the other hand, h is not an isomorphism, since $h(ab) = h(b)$.

Corollary 4.10. *Let $H \vdash I : \mathbb{C} \to \mathbb{M}$ be a simple reflection of a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$, such that for all*

$s, s' \in S$, where $S \in \mathbb{C}$ if $s \sim_S s'$ then there exist $w, w' \in FU(S)$, such that $w \sim_{FU(S)} w'$, $\varepsilon_S(w) = s$, $\varepsilon_S(w') = s'$, where $\varepsilon : FU \to \mathbf{1}_{\mathbb{C}}$ is the counit of the adjunction $\langle F, U, \lambda, \varepsilon \rangle : \mathbf{Set} \to \mathbb{C}$.

Suppose that in the following pullback diagram $\pi_1 \in \mathcal{M}_I$.

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & L \arrow[d, "f"] \\ FU(S) \arrow[r, "\varepsilon_S"] & S \end{tikzcd} \tag{4.12}$$

Then, the following conditions are equivalent:

(1) *$I(\pi_1)$, $I(\pi_2)$ are jointly monic;*

(2) *the reflector I preserves the pullback* (4.12)*.*

Proof. If I preserves the pullback (4.12) then, $I(\pi_1)$, $I(\pi_2)$ are jointly monic.

Conversely, consider the pullback diagram (4.8) and let $w \in FU(S)$ and $t \in L$ be such that $\varepsilon_S(w) \sim_S t$. Then, there exist $w_1, w_2 \in FU(S)$, such that $w_1 \sim_{FU(S)} w_2$, $\varepsilon_S(w_1) = \varepsilon_S(w)$, $\varepsilon_S(w_2) = f(t)$.

Since $\pi_1 \in \mathcal{M}_I$, $(w_2, t) \in FU(S) \times_S L$ and $w_1 \sim_{FU(S)} w_2$, there exists $t^* \in L$ such that $(w_1, t^*) \in FU(S) \times_S L$ and $(w_1, t^*) \sim_{FU(S) \times_S L} (w_2, t)$. Hence, $t^* \sim_L t$.

On the other hand, since $\varepsilon_S(w) = \varepsilon_S(w_1) = f(t^*)$, $(w, t^*) \in FU(S) \times_S L$. Therefore, α is surjective.

$\square$

Remark 4.11. Consider the reflection $H \vdash I : \mathbf{Band} \to \mathbf{SLat}$.

In this reflection $\mathcal{M}^*{}_I = \mathcal{M}_I$.

In order to prove it let A be a band and let $a,\ b \in A$.

Recall that $a \sim_A b$ if and only if $a = aba,\ b = bab$.

The elements $aba,\ bab$ are in the same rectangular band of $FU(A)$ and $\varepsilon_A(aba) = a,\ \varepsilon_A(bab) = b$.

By Example 3.17, the reflection of the projections $I(\pi_1)$ and $I(\pi_2)$

of any pullback are jointly monic.

Then, by Corollary 4.10 and Proposition 4.8 the Galois groupoid of $(A,\ f)$ for any Galois descent homomorphism $f : A \to B$ in **Band** is the equivalence relation given by the kernel pair of $I(f)$ in **SLat** and $\mathcal{M}^*{}_I = \mathcal{M}_I$.

The following Example 4.12 exhibits a Galois descent homomorphism (which is not an isomorphism), for the reflection $H \vdash I$: **Band** $\to$ **SLat**.

EXAMPLE 4.12. Since for the reflection $H \vdash I$: **Band** $\to$ **SLat**, $I(\pi_1)$ and $I(\pi_2)$, in the pullback (4.12), are jointly monic, by Example 3.17, $\mathcal{M}^*{}_I = \mathcal{M}_I$. Hence, a homomorphism is a Galois descent if and only if it belongs to $\mathcal{M}_I \cap \mathcal{E}$.

For instance, consider the band R of Example 3.6 and the band S of Example 3.4 and the surjective homomorphism $g : R \to S,\ g(a) = a;\ g(b) = b;\ g(r) = a;\ g(q) = b$. Since R is a semilattice of two isomorphic rectangular bands, isomorphic to the semilattice indecomposable band S and the restrictions of g to each one of the classes of R are isomorphisms, $g \in \mathcal{M}_I \cap \mathcal{E}$.

On the other hand, obviously, g is not an isomorphism.

4.2. Covering morphisms in a more general setting

The next Proposition 4.14 generalizes Corollary 4.10 for simple reflections $H \vdash I : \mathbb{A} \to \mathbb{B}$ of a finitely complete category $\mathbb{A}$ into a full subcategory $\mathbb{B}$.

Consider the following data (4.13):

(1) Let $\mathbb{A}$ be a finitely complete category, and let $\mathbb{B}$ be a full reflective subcategory of $\mathbb{A}$, the inclusion functor being $H : \mathbb{B} \to \mathbb{A}$, with unit $\eta : \mathbf{1}_{\mathbb{A}} \to HI$.

(2) Let $U : \mathbb{A} \to$ **Set** be a functor that satisfies the following conditions (a), (b), (c):

(a) $U(\eta_A)$ is surjective, for all $A \in \mathbb{A}$;

(b) U preserves finite limits;

(c) U reflects isomorphism.

The following Remark 4.13 is done in order to see certain property of some reflections, that will be used forward, in terms of kernel pairs of the unit morphisms.

Remark 4.13. Let $H \vdash I : \mathbb{A} \to \mathbb{B}$ be a reflection of a category $\mathbb{A}$, with pullbacks and unit $\eta : \mathbf{1}_{\mathbb{A}} \to HI$, into a subcategory $\mathbb{B}$, and let $U : \mathbb{A} \to \mathbf{Set}$ be a functor that preserves finite limits. Then, the following conditions are equivalent:

- for all $b, b' \in U(B)$, if $U(\eta_B)(b) = U(\eta_B)(b')$ then, there exist $a, a' \in U(A)$ such that $U(f)(a) = b$, $U(f)(a') = b'$ and $U(\eta_A)(a) = U(\eta_A)(a')$;
- in the following pullback $U(w)$ is a surjection, where $w = \langle f \circ p_1,\ f \circ p_2 \rangle$.

(4.14)
$$\begin{tikzcd}
A \times_{HI(A)} A \arrow[rrr, "p_2"] \arrow[dr, "w"] \arrow[ddd, "p_1"'] & & & A \arrow[dl, "f"'] \arrow[ddd, "\eta_A"] \\
& B \times_{HI(B)} B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "\eta_B"] & \\
& B \arrow[r, "\eta_B"] & HI(B) & \\
A \arrow[ur, "f"] \arrow[rrr, "\eta_A"'] & & & HI(A) \arrow[ul, "HI(f)"']
\end{tikzcd}$$

Proposition 4.14. *Assume data* (4.13) *and suppose that:*

- *the reflection is simple;*
- *concerning the morphism $g : A \to S$, in $\mathbb{A}$, for all $s, s' \in U(S)$, which verify $U(\eta_S)(s) = U(\eta_S)(s')$, there exist $w, w' \in U(A)$, such that $U(\eta_A)(w) = U(\eta_A)(w')$, $U(g)(w) = s$, $U(g)(w') = s'$;*

- *in the following pullback diagram* $\pi_1 \in \mathcal{M}_I$.

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & L \arrow[d, "f"] \\ A \arrow[r, "g"] & S \end{tikzcd} \tag{4.15}$$

Then, the following conditions are equivalent:

(1) $I(\pi_1)$ *and* $I(\pi_2)$ *are jointly monic;*

(2) *the pullback* (4.15) *is preserved by the reflector* I.

Proof. If the reflector I preserves the pullback (4.15), then $I(\pi_1)$ and $I(\pi_2)$ are jointly monic.

Conversely, we want to prove that the morphism α in the following pullback diagram is an isomorphism.

$$\begin{tikzcd} HI(A \times_S L) \arrow[dr, "\alpha"] \arrow[drr, "HI(\pi_2)"] \arrow[ddr, "HI(\pi_1)"'] & & \\ & HI(A) \times_{HI(S)} HI(L) \arrow[r] \arrow[d] & HI(L) \arrow[d, "HI(f)"] \\ & HI(A) \arrow[r, "HI(g)"] & HI(S) \end{tikzcd} \tag{4.16}$$

Since $I(\pi_1)$ and $I(\pi_2)$ are jointly monic and U preserves finite limits, $U(\alpha)$ is an injective map. Hence, it remains to prove that $U(\alpha)$ is, also, surjective.

Let $w \in U(A)$ and $t \in U(L)$ be such that $U(\eta_S) \circ U(f)(t) = U(\eta_S) \circ U(g)(w)$.
If there exists $(w', t') \in U(A) \times_{U(S)} U(L)$, such that $U(\eta_L)(t') = U(\eta_L(t)$ and $U(\eta_A)(w) = U(\eta_A)(w')$, then $U(\alpha)$ is a surjection.

Since $U(\eta_S) \circ U(f)(t) = U(\eta_S) \circ U(g)(w)$, by hypothesis, there exist $w_1, w_2 \in U(A)$ such that $U(\eta_A)(w_1) = U(\eta_A)(w_2)$, $U(g)(w_1) = U(g)(w)$, $U(g)(w_2) = U(f)(t)$. Hence, $(w_2, t) \in U(A) \times_{U(S)} U(L)$.

Consider the following pullback diagram

$$
\begin{tikzcd}
U(A) \times_{U(S)} U(L) \arrow[dr, "\phi"] \arrow[drr, "U(\eta_{A\times_S L})"] \arrow[ddr, "U(\pi_1)"'] & & \\
& P \arrow[r] \arrow[d] & UHI(A \times_S L) \arrow[d, "UHI(\pi_1)"] \\
& U(A) \arrow[r, "U(\eta_A)"] & UHI(A)
\end{tikzcd}
\quad , \tag{4.17}
$$

where $P = U(A) \times_{UHI(A)} UHI(A \times_S L)$.

Since $\pi_1 \in \mathcal{M}_I$, ϕ is a bijection.

Consider the following commutative square:

$$
\begin{tikzcd}
U(A \times_S L) \arrow[r, "U(\eta_{A\times_S L})"] \arrow[d, "U(\pi_1)"'] & UHI((A) \times_S L) \arrow[d, "UHI(\pi_1)"] \\
U(A) \arrow[r, "U(\eta_A)"] & UHI(A)
\end{tikzcd}
\tag{4.18}
$$

Since $U(\eta_A) \circ U(\pi_1)(w_2, t) = UHI(\pi_1) \circ U(\eta_{A\times_S L})(w_2, t)$, that is, $U(\eta_A)(w_2) = UHI(\pi_1) \circ U(\eta_{A\times_S L})(w_2, t)$ and $U(\eta_A)(w_2) = U(\eta_A)(w_1)$, $(w_1, U(\eta_{A\times_S L})(w_2, t)) \in U(A \times_{HI(A)} HI(A \times_S L))$.

Since ϕ is surjective, there exists $(w', t') \in U(A \times_S L)$, such that $w' = w_1$ and $U(\eta_{A\times_S L})(w', t') = U(\eta_{A\times_S L})(w_2, t)$.

Consider the following commutative square:

$$
\begin{tikzcd}
U(A \times_S L) \arrow[r, "U(\eta_{A\times_S L})"] \arrow[d, "U(\pi_2)"'] & UHI(A \times_S L) \arrow[d, "UHI(\pi_2)"] \\
U(L) \arrow[r, "U(\eta_L)"] & UHI(L)
\end{tikzcd}
\tag{4.19}
$$

Since $U(\eta_{A\times_S L})(w_1, t') = U(\eta_{A\times_S L})(w_2, t)$, $U(\eta_L)(t') = U(\eta_L)(t)$.

On the other hand, $U(g)(w_1) = U(g)(w)$. Hence, $(w, t') \in U(A) \times_{U(S)} U(L)$ and $U(\eta_L)(t) = U(\eta_L)(t')$. Therefore, $U(\alpha)$ is a bijection.

Since U reflects isomorphisms, the pullback (4.15) is preserved by I.

□

Corollary 4.15. *Under the equivalent conditions (1) and (2) of Proposition 4.14, if in the pullback* (4.15) *$g : A \to S$ is an effective descent morphism in $\mathbb{A}$, then $f : L \to S \in \mathcal{M}_I$ if and only if $\pi_1 \in \mathcal{M}_I$.*

Proof. It is just the proof of Corollary 4.6. □

Proposition 4.16. *Under the equivalent conditions (1) and (2) of Proposition 4.14, if the reflection is semi-left-exact and if the morphisms f and g in the pullback* (4.15) *are such that:*

- *f is a Galois descent morphism in $\mathbb{A}$;*
- *g is an effective descent morphism in $\mathbb{A}$;*
- *there exists $m : A \to L$ in $\mathbb{A}$, such that $g = f \circ m$.*

Then

- *the Galois groupoid of $(L,\ f)$ is the equivalence relation given by the kernel pair of $I(f)$, in $\mathbb{B}$;*
- $\mathcal{M}_I/S \cong \mathbb{B}^{Gal[f]}$.

Proof. Since f is a Galois descent morphism in $\mathbb{A}$, its pullback along f, $f^*(f)$ belongs to $\mathcal{M}_I$. Hence, the pullback of f along $g = f \circ m$ also belongs to $\mathcal{M}_I$. Thus, by Corollary 4.15, $f \in \mathcal{M}_I$. Since the reflection is semi-left-exact, I preserves the kernel-pair of f, by Proposition

1.7.

On the other hand $\mathcal{M}_I/S = \mathbf{Split}_S(f)$, by Corollary 4.15. The equivalence of categories follows from Theorem 1.31.

□

EXAMPLE 4.17. Consider a semi-left-exact reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$, of a category $\mathbb{C}$ into a full subcategory $\mathbb{M}$, such that there exists an adjunction $\langle F, U, \lambda, \varepsilon \rangle : \mathbf{Set} \to \mathbb{C}$, with unit λ, and counit ε, such that:

(1) ε_B is an effective descent morphism in $\mathbb{C}$, for all $B \in \mathbb{C}$;

(2) if p is an effective descent morphism in $\mathbb{C}$ then $U(p)$ is a split epimorphism in **Set**.

Let $g = \varepsilon_S : FU(S) \to S$ in the pullback (4.15) of Proposition 4.14.

Then, by Lemma 4.2, there exists $m : FU(S) \to L$ in $\mathbb{C}$, such that $\varepsilon_S = f \circ m$.

CHAPTER 5

Separable morphisms

In this chapter we will describe the classes of separable, purely inseparable and normal homomorphisms, for reflections $H \vdash I : \mathbb{C} \to \mathbb{M}$ from a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$. We conclude that, when the class of stably-vertical homomorphism is $\mathcal{E}'_I = \mathcal{E}_I \cap \mathcal{F}$, there is an Inseparable-Separable factorization. For instance, in the reflection of bands into semilattices and in the reflection of commutative semigroups into semilattices.

5.1. Separable, purely inseparable and normal morphisms

In this section we will describe the classes of separable, purely inseparable and normal homomorphisms, for reflections $H \vdash I : \mathbb{C} \to \mathbb{M}$ from a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$.

Consider a reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ from a variety $\mathbb{C}$ of universal algebras into a subvariety $\mathbb{M}$.

Definition 5.1. Consider the following pullback diagram

(5.1)
$$\begin{tikzcd}
A \arrow[dr, "\delta_\alpha"] \arrow[drr, "1_A"] \arrow[ddr, "1_A"'] & & \\
& A \times_B A \arrow[r, "v"] \arrow[d, "u"] & A \arrow[d, "\alpha"] \\
& A \arrow[r, "\alpha"] & B
\end{tikzcd},$$

where (u, v) is the kernel-pair of the homomorphism $\alpha : A \to B$.

With respect to the reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ from a variety $\mathbb{C}$ of universal algebras into a subvariety $\mathbb{M}$:

- α is called a separable homomorphism ($\alpha \in \mathbf{Sep}$) if δ_α is a trivial covering, i.e., $\delta_\alpha \in \mathcal{M}_I$.
- α is called a purely inseparable homomorphism ($\alpha \in \mathbf{Pin}$) if δ_α is vertical, i.e., $\delta_\alpha \in \mathcal{E}_I$.
- α is called a normal homomorphism ($\alpha \in \mathbf{Normal}$) if u is a trivial covering, i.e., $u \in \mathcal{M}_I$.

Proposition 5.2. *A homomorphism $\alpha : A \to B$ is separable if and only if for all $a, a' \in A$, such that $\alpha(a) = \alpha(a')$, if $(a, a') \sim_{A\times_B A} (d, d)$, for some $d \in A$, then $a = a'$.*

In other words, a homomorphism $\alpha : A \to B$ is separable if and only if in the kernel pair of α, $Ker(\alpha) = A \times_B A$, a congruence class either is constituted only by ordered pairs which have equal components or none of its elements is an ordered pair with equal components.

Proof. A homomorphism $\alpha : A \to B$ is separable if and only if, for all $a \in A$, (5.2) is a bijection.

$$\delta_{\alpha|[a]_{\sim_A}} : [a]_{\sim_A} \to [\delta_\alpha(a)]_{\sim_{A\times_B A}} \tag{5.2}$$

is always injective. Since $v \circ \delta_\alpha = 1_A$, δ_α is a split mono and then, is an injective homomorphism. Hence,the restrictions (5.2are injective.

On the other hand, (5.2) is surjective if and only if for all $a, a' \in A$, such that $\alpha(a) = \alpha(a')$, if $(a, a') \sim_{A\times_B A} (d, d)$, for some $d \in A$ then, $a = a'$. □

Proposition 5.3. *A homomorphism $\alpha : A \to B$ is purely inseparable if and only if for every $a, a' \in A$, such that $\alpha(a) = \alpha(a')$, there exists $c \in A$, such that $(c, c) \sim_{A\times_B A} (a, a')$.*

In other words, a homomorphism $\alpha : A \to B$ is purely inseparable if and only if in every congruence class of the kernel pair of α,

$Ker(\alpha) = A \times_B A$, there exists at least one element which is an ordered pair with equal components.

PROOF. A homomorphism $\alpha : A \to B$ is purely inseparable if and only if $I(\delta_\alpha)$ is an isomorphism.

Since $v \circ \delta_\alpha = 1_A$, $I(v) \circ I(\delta_\alpha) = 1_{I(A)}$. Hence, $I(\delta_\alpha)$ is always injective. On the other hand, $I(\delta_\alpha)$ is surjective if and only if for every $a, a' \in A$, such that $\alpha(a) = \alpha(a')$, there exists $c \in A$, such that $(c, c) \sim_{A \times_B A} (a, a')$. $\square$

PROPOSITION 5.4. *A homomorphism $\alpha : A \to B$ is normal if and only if the next two conditions hold:*

(1) *if $\alpha(a_1) = \alpha(a_2)$, and there exists $a \in A$ such that $a \sim_A a_1$ then, there exists $b \in A$, such that $\alpha(a) = \alpha(b)$, and $(a_1, a_2) \sim_{A \times_B A} (a, b)$ or, equivalently, $(a_2, a_1) \sim_{A \times_B A} (b, a)$, for all $a_1, a_2 \in A$*

(2) *if $\alpha(c) = \alpha(d_1)$, $\alpha(c) = \alpha(d_2)$, and $(c, d_1) \sim_{A \times_B A} (c, d_2)$ or, equivalently, $(d_1, c) \sim_{A \times_B A} (d_2, c)$ then, $d_1 = d_2$, for all $c, d_1, d_2 \in A$.*

PROOF. Recall that $u \in \mathcal{M}_I$, or equivalently, $v \in \mathcal{M}_I$, if and only if $u_{|[(a,a')]_{\sim_{A \times_B A}}} : [(a, a')]_{\sim_{A \times_B A}} \to [u(a, a')]_{\sim_A}$ is a bijection, for all $(a, a') \in A \times_B A$.

Condition (1) is equivalent to say that those restrictions are surjective, while (2) is equivalent to say that they are injective. $\square$

COROLLARY 5.5. *If $I(u)$ and $I(v)$ are jointly monic in Definition 5.1, for every Kernel pair (u, v), then :*

- *A homomorphism $\alpha : A \to B$ is separable if and only if $Ker(\alpha) \cap \sim_A = \Delta$.*
- *A homomorphism $\alpha : A \to B$ is purely inseparable if and only if $Ker(\alpha) \subseteq \sim_A$.*
- *A homomorphism $\alpha : A \to B$ is normal if and only if the next two condition hold:*

(1) $\sim_A \circ Ker(\alpha) \subseteq Ker(\alpha) \circ \sim_A$,

(2) $Ker(\alpha) \cap \sim_A = \Delta$.

Where Δ *denotes the equality relation,* $Ker(\alpha)$ *denotes the kernel pair of* α, $\sim_A$ *denotes the congruence on* A *induced by the reflection and* $\circ$ *denotes the composition of congruences.*

PROOF.

- A homomorphism $\alpha : A \to B$ is separable if and only if, for all $a, a' \in A$, such that $\alpha(a) = \alpha(a')$, and there exists $b \in A$ such that $b \sim_A a$, $b \sim_A a'$ then, there exists $c \in A$, such that $(c, c) = (a, a')$. That is, a homomorphism $\alpha : A \to B$ is separable if and only if, for all $a, a' \in A$, such that $\alpha(a) = \alpha(a')$, and $a \sim_A a'$, then, $a = a'$. Which is to say that a homomorphism $\alpha : A \to B$ is separable if and only if $Ker(\alpha) \cap \sim_A = \Delta$.

- A homomorphism $\alpha : A \to B$ is purely inseparable if and only if if for all $a, a' \in A$, if $\alpha(a) = \alpha(a')$ then, there exists $b \in A$ such that $b \sim_A a$, $b \sim_A a'$. That is, a homomorphism $\alpha : A \to B$ is purely inseparable if and only if if for all $a, a' \in A$, if $\alpha(a) = \alpha(a')$ then, $a \sim_A a'$. Which is to say that a homomorphism $\alpha : A \to B$ is separable if and only if $Ker(\alpha) \subseteq \sim_A$.

- A homomorphism $\alpha : A \to B$ is normal if and only if the next two condition hold:

  (1) The restrictions of u to the congruence classes of $Ker(\alpha)$ are surjective if and only if for all a, $b \in A$ such that there exists $c \in A$ with $a \sim_A c$ and $\alpha(c) = \alpha(b)$ then, there exists $d \in A$, such that $\alpha(a) = \alpha(d)$ and $d \sim_A b$. That is, $\sim_A \circ Ker(\alpha) \subseteq Ker(\alpha) \circ \sim_A$.

  (2) The restrictions of u to the congruence classes of $Ker(\alpha)$ are injective if and only if for all a, $b \in A$ such that $\alpha(a) = \alpha(b)$ and $a \sim_A b$ then, $a = b$. That is, $Ker(\alpha) \cap \sim_A = \Delta$.

$\square$

EXAMPLE 5.6. This characterization holds in both reflections **CommSgr** into **SLat** and **Band** into **SLat**.

Under the conditions of Proposition 4.5 or under the conditions of Corollary 4.10, if in the following pullback diagram,

$$
\begin{tikzcd}
P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & L \arrow[d, "p"] \\
L \arrow[r, "p"] & S
\end{tikzcd}
\qquad ,
\tag{5.3}
$$

$I(\pi_1)$ and $I(\pi_2)$ are jointly monic, $\pi_1 \in \mathcal{M}_I$ and p is an effective descent homomorphism, then $p \in \mathcal{M}_I$.

It is known that for a variety of universal algebras the effective descent morphisms are the surjective homomorphisms. Hence, if $p : S \to L$ is an effective descent morphism then, by Lemma 4.2, there exists a homomorphism $f : FU(L) \to S$, such that, $p \circ f = \varepsilon_L$. Since $\pi_1 \in \mathcal{M}_I$ then, the pullback of p along $p \circ f = \varepsilon_L$ belongs to $\mathcal{M}_I$. By Corollary 4.6, $p \in \mathcal{M}_I$.

On the other hand, in the next Examples 5.7 (3), 5.8 (3), we will exhibit some normal homomorphisms which do not belong to $\mathcal{M}_I$, while we are under the conditions of Proposition 4.5 or under the conditions of Corollary 4.10.

Suppose that $(\mathcal{E}_I, \mathcal{M}_I)$ is a prefactorization system. If a morphism $\alpha : A \to B$ is normal, then it is separable.

Consider the pullback (5.1). Since $u \circ \delta_\alpha = 1_A$ and $u,\ 1_A \in \mathcal{M}_I$ then, $\delta_\alpha \in \mathcal{M}_I$, by Proposition 1.1.

On the other hand, in the next Examples 5.7 (1), 5.8 (1), we will exhibit some separable homomorphisms which are not normal.

Example 5.7.

Consider the reflection $H \vdash I$: **Band** $\to$ **SLat**. By Example 3.17, $I(u)$ and $I(v)$ are jointly monic Definition 5.1, for every Kernel pair (u, v). Hence, the classes **Sep**, **Pin** and **Normal** are just those described in Remark 5.5.

(1) The following homomorphism is a separable homomorphism which is not a normal homomorphism (this homomorphism is given in [**9**]).

Consider the band B given by the following table, the band S in the Example 3.4 and the homomorphism $h : B \to S,\ h(a) = h(q) = a,\ h(b) = h(r) = b.$

(5.4)

$\bullet$	a	b	r	q
a	a	a	a	a
b	b	b	b	b
r	a	a	r	a
q	b	b	b	q

The semilattice classes of B are $\{a,\ b\},\ \{r\},\ \{q\}$ and S is semilattice indecomposable. It is clear that $Ker(h) \cap \sim_B = \Delta$. Hence, $h \in$ **Sep**. On the other hand, $(b, q) \in \sim_S \circ Ker(h)$, since a is such that $(b, a) \in \sim_S$ and $(a, q) \in Ker(h)$, but there does not exist $x \in B$, such that $h(b) = h(x)$ and $x \sim_B a$, then $(b, q) \notin Ker(h) \circ \sim_B$. Therefore, $\sim_B \circ Ker(h)$ is not contained in $Ker(h) \circ \sim_B$. Hence, h is not a normal homomorphism.

(2) The following homomorphism is a purely inseparable one. Consider the band S in the Example 3.4. The homomorphism $h : S \to S,\ h(a) = h(b) = a$ is purely inseparable, since $Ker(h) \subseteq \sim_S$.

(3) The following homomorphism is a normal homomorphism which does not belong to $\mathcal{M}_I$. Consider the band R of Example 3.6, the band L given by the following table

(5.5)

$\bullet$	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

The band R is the semilattice of the rectangular bands $\{a,\ b\},\ \{q,\ r\}$, while L is a rectangular band, since $x = xyx$, for every $x,\ y \in L$.

The homomorphism $h : R \to L;\ h(a) = h(q) = a;\ h(q) = h(r) = b$ is a normal homomorphism, since $\sim_R \circ Ker(h) \subseteq$

$Ker(h) \circ \sim_R$ and $Ker(h) \cap \sim_R = \Delta$. On the other hand, the restriction of h to any class is not an isomorphism, so $h \notin \mathcal{M}_I$.

EXAMPLE 5.8.
Consider the reflection $H \vdash I : \mathbf{CommSgr} \rightarrow \mathbf{SLat}$.

By Example 3.12, $I(u)$ and $I(v)$ are jointly monic in Definition 5.1, for every Kernel pair (u, v). Hence, the classes **Sep**, **Pin** and **Normal** are just those described in Remark 5.5.

(1) The following homomorphism is a separable homomorphism which is not a normal homomorphism. Let $S = F(a,\ b;\ a = a^2;\ b^2 = b^3;\ b^2 = ab)$ and $R = F(b;\ b^2 = b^3)$. The commutative semigroup S is a semilattice of two archimedean classes $\{a\},\ \{b,\ b^2\}$, while R is archimedean with one idempotent b^2. Let $h : S \rightarrow R$ be the homomorphism given by $h(a) = b^2;\ h(b) = b$. The homomorphism h is separable, since $Ker(h) \cap \sim_R = \Delta$. On the other hand, $(b, a) \in \sim_S \circ ker(h)$, since b^2 is such that $(b, b^2) \in \sim_S$ and $(b^2, a) \in ker(h)$, but there does not exist $x \in S$, such that $h(b) = h(x)$ and $x \sim_S a$, then $(b, a) \notin Ker(h) \circ \sim_S$. Therefore, $\sim_S \circ Ker(h)$ is not contained in $Ker(h) \circ \sim_S$, that is, $h \notin$ **Normal**.

(2) The following homomorphism is a purely inseparable one. Let $(\mathbf{Z}, *),\ (\mathbf{N}, *)$ be, respectively, the commutative semigroup of integer and natural numbers with the usual multiplication. Let $|-| : \mathbf{Z} \rightarrow \mathbf{N}$ be the homomorphism $|z| = z$, if $z \geq 0$; $|z| = -z$, if $z \leq 0$. Clearly, $Ker(h) = \{(z, z),\ (z, -z) \mid z \in \mathbf{Z}\}$. On the other hand $z = (-1)(-z),\ (-z) = (-1)z$, for all $z \in \mathbf{Z}$. Hence, $Ker(h) \subseteq \sim_{(\mathbf{Z},*)}$. Therefore, $h \in$ **Pin**.

(3) The following homomorphism is a normal homomorphism which does not belong to $\mathcal{M}_I$. Let $h : \mathbf{N} \rightarrow \mathbf{Z}$ be the inclusion homomorphism of the additive semigroup of natural numbers into the additive semigroup of rational numbers, both archimedean, hence with only one component in the semillatice decomposition. Clearly, $Ker(h) \cap \sim_{\mathbf{N}} = \Delta$ and $\sim_{\mathbf{N}} \circ Ker(h) \subseteq Ker(h) \circ \sim_{\mathbf{N}}$, then h is a normal homomorphism. On the other hand, h is not an isomorphism then, $h \notin \mathcal{M}_I$.

5.2. Inseparable-Separable factorization

In this section we will show that there exists an (**Ins**, **Sep**) factorization system under the equivalent conditions of Proposition 3.10.

Consider a simple reflection $H \vdash I : \mathbb{C} \to \mathbb{M}$ from a variety of universal algebras $\mathbb{C}$ into a subvariety $\mathbb{M}$. For the factorization system $(\mathcal{E}_I, \mathcal{M}_I)$ induced by the reflection, we can form a derived factorization system (**Ins**, **Sep**), if and only if the class **Ins** of inseparable homomorphisms is closed under composition (see the following Lemma 5.11).

The (**Ins**, **Sep**)-factorization of a morphism $\alpha : A \to B$ is given in the following diagram (cf. §1.8., [**16**]):

(5.6)
$$\begin{tikzcd}
& {} \arrow[dr, "m_{\delta_\alpha}"] & & & {} \arrow[dr, "m^*_\alpha"] & \\
A \arrow[ur, "e_{\delta_\alpha}"] \arrow[rr, "\delta_\alpha"'] & & A \times_B A \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & A \arrow[ur, "e'_\alpha"] \arrow[rr, "\alpha"'] & & B,
\end{tikzcd}$$

where:

- e'_α is the coequalizer of $(u \circ m_{\delta_\alpha}, v \circ m_{\delta_\alpha})$,
- (u, v) is the kernel-pair of α,
- $m_{\delta_\alpha} \circ e_{\delta_\alpha}$ is the $(\mathcal{E}_I, \mathcal{M}_I)$-factorization of the homomorphism δ_α of diagram (5.1).

Definition 5.9. A morphism $\alpha : A \to B$ is called inseparable with respect to the reflection $H \vdash I : A \to B$ if, in its factorization $\alpha = m^*_\alpha e'_\alpha$ given in diagram (5.6), m^*_α is an isomorphism.

Let **Sep**, **Pin**, and **Ins** denote the classes of separable morphisms, purely inseparable morphisms and inseparable morphisms, respectively.

The statement and proof of next Lemmas 5.10 and 5.11 can be found in [**8**, §3,§4].

Lemma 5.10. *For the factorization given at diagram (5.1), the following two inclusions and one equality hold, where RegEpi is the class of regular epimorphisms in $\mathbb{C}$ (which for varieties of universal algebras are the surjective homomorphisms, denoted by $\mathcal{E}$).*

- $\mathbf{Pin} \cap \mathit{RegEpi} \subseteq \mathbf{Ins} \subseteq \mathcal{E}_I \cap \mathit{RegEpi}$,
- $\mathbf{Sep} = \{\alpha \mid e'_\alpha \text{ is an isomorphism}\}$.

LEMMA 5.11. *The pair* $(\mathbf{Ins}, \mathbf{Sep})$ *is a factorization system if and only if the class of morphisms* $\mathbf{Ins}$ *is closed under composition.*

PROPOSITION 5.12. *Under the equivalent conditions of Proposition* 3.10*, there exists a factorization system* $(\mathbf{Ins}, \mathbf{Sep})$, *with* $\mathbf{Ins} = \mathcal{E}_I \cap \mathcal{E}$.

PROOF. First notice that $\mathcal{E}'_I \subseteq \mathbf{Pin}$:

Consider the pullback diagram (5.1) and suppose $\alpha \in \mathcal{E}'_I$ and then, $v \in \mathcal{E}_I$. Since $I(v)$ is an isomorphism and $I(v) \circ I(\delta_\alpha) = 1_{I(A)}$, $I(\delta_\alpha)$ is an isomorphism. Hence, $\delta_\alpha \in \mathcal{E}_I$, thus $\alpha \in \mathbf{Pin}$.

Therefore, by Lemma 5.10,

$$\mathcal{E}'_I \cap \mathcal{E} \subseteq \mathbf{Ins} \subseteq \mathcal{E}_I \cap \mathcal{E}.$$

Since $\mathcal{E} \subseteq \mathcal{F}$, $\mathcal{E}_I \cap \mathcal{E} \subseteq \mathcal{E}_I \cap \mathcal{F} = \mathcal{E}'_I$. Hence, $\mathcal{E}_I \cap \mathcal{E} \subseteq \mathcal{E}_I \cap \mathcal{F} \cap \mathcal{E} = \mathcal{E}'_I \cap \mathcal{E}$. On the other hand, since $\mathcal{E}'_I \subseteq \mathcal{E}_I$, $\mathcal{E}'_I \cap \mathcal{E} \subseteq \mathcal{E}_I \cap \mathcal{E}$. Hence, $\mathcal{E}'_I \cap \mathcal{E} = \mathcal{E}_I \cap \mathcal{E}$. Therefore, $\mathbf{Ins} = \mathcal{E}_I \cap \mathcal{E}$.

Since $\mathcal{E}_I$ and $\mathcal{E}$ are closed under composition, so it is $\mathcal{E}_I \cap \mathcal{E}$. Then, by Lemma 5.11, $(\mathbf{Ins}, \mathbf{Sep})$ is a factorization system.

□

EXAMPLE 5.13. Consider the reflection $H \vdash I : \mathbf{CommSgr} \to \mathbf{SLat}$. By Example 3.12 of Proposition 3.10, $\mathcal{E}'_I = \mathcal{E}_I \cap \mathcal{F}$. Thus, we are under the conditions of Proposition 5.12. Therefore, for this reflection, there exists a factorization system $(\mathbf{Ins}, \mathbf{Sep})$, with $\mathbf{Ins} = \mathcal{E}_I \cap \mathcal{E}$.

Consider the reflection $H \vdash I : \mathbf{Band} \to \mathbf{SLat}$. By Example 3.17 of Corollary 3.16, $\mathcal{E}'_I = \mathcal{E}_I \cap \mathcal{E}$. Thus, we are under the conditions of Proposition 5.12, since $\mathcal{E} = \mathcal{F}$ for $\mathbf{Band}$. Therefore, for this reflection, there exists a factorization system $(\mathbf{Ins}, \mathbf{Sep})$, with $\mathbf{Ins} = \mathcal{E}_I \cap \mathcal{E}$.

The following Remark 5.14 gives an obvious sufficient condition for the reflection of Proposition 3.20 to have an ($\mathbf{Ins}$, $\mathbf{Sep}$) factorization system.

REMARK 5.14. Consider data (3.24), suppose that $\mathbb{A}$ has coequalizers, $(\mathcal{E}_I, \mathcal{M}_I)$ is a factorization system, for instance if the reflection is simple, and $\mathit{RegEpi} \subseteq \mathcal{G}$. Then, under the conditions of Proposition 3.20 ($\mathbf{Ins}$, $\mathbf{Sep}$), is a factorization system, with $\mathbf{Ins} = \mathcal{E}_I \cap \mathit{RegEpi}$.

In the following Remark 5.15 we conclude that in the reflection $H \vdash I : \mathbf{Band} \to \mathbf{SLat}$ there is no monotone-light factorization system.

REMARK 5.15. Recall that in the reflection of bands into semilattices,

(1) There exists a factorization system $(\mathbf{Ins}, \mathbf{Sep})$, with $\mathbf{Ins} = \mathcal{E}_I \cap \mathcal{E}$.

(2) $\mathbf{Ins} = \mathcal{E}'_I = \mathcal{E}_I \cap \mathcal{E}$,

(3) $\mathcal{M}^*{}_I = \mathcal{M}_I$,

On one hand, the monomorphisms of bands are, clearly, separable. Let $\alpha : A \to B$ be an injective homomorphism, then $ker(\alpha) = \Delta$. Hence, $Ker(\alpha) \cap \sim_A = \Delta$.

On the other hand, the inclusion homomorphism of the band S given by the following table

(5.7)

$\bullet$	a	b
a	a	a
b	b	b

into the band L given by the following table

(5.8)

$\bullet$	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

is a monomorphism which does not belong to $\mathcal{M}_I = \mathcal{M}^*{}_I$.

Since $\mathbf{Ins} = \mathcal{E}'_I$ and $\mathbf{Sep} \neq \mathcal{M}^*{}_I$, there is no monotone-light factorization.

In the following Remark 5.16 we conclude that in the reflection $H \vdash I : \mathbf{CommSgr} \to \mathbf{SLat}$ there does not exist a monotone-light factorization system.

REMARK 5.16. Recall that in the reflection of commutative semigroups into semilattices,

(1) There exists a factorization system $(\mathcal{E}_I, \mathcal{M}_I)$, with $\mathcal{E}_I \neq \mathcal{E}'_I$, by Remark 3.13.

(2) On the other hand, $\mathcal{M}^*{}_I = \mathcal{M}_I$, by Example 4.7.

Therefore this reflection does not have a monotone-light factorization system.

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